Functions of trigonometric type and bases in L_q

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Abstract

It is shown that, for all $p \in (1, \infty)$, the eigenfunctions of the Dirichlet problem for the p-Laplacian on $[0, 1]$ form a basis of $L_q(0, 1)$ for all $q \in (1, \infty)$.

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1 Introduction

Let $q \in (1, \infty)$. It is a standard fact that the functions $\sin n\pi x$ and $\cos n\pi x$ form a basis of $L_q(-1, 1)$: see, for example, [6], pp. 342-5. Given any $f \in L_q(0, 1)$, it follows that its odd extension to $L_q(-1, 1)$ has a unique representation in terms of the $\sin n\pi x$. This means that the $\sin n\pi x$ form a basis of $L_q(0, 1)$. In this paper we show that the same is true when the sines are replaced by the p-sine functions, for any $p \in (1, \infty)$. We recall that these may

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be defined by setting

$$F_p(x) = \int_0^x (1-t^p)^{-1/p} dt, \qquad x \in [0,1] \tag{1.1}$$

and

$$\pi_p = 2\int_0^1 (1-t^p)^{-1/p} dt. \tag{1.2}$$

The p-sine function, $\sin_p$, is defined on $[0, \pi_p/2]$ by

$$\sin_p x = F_p^{-1}(x); \tag{1.3}$$

it is extended to $\mathbf{R}$ by standard procedures. Note that $\sin_2$ is simply the usual sine function. These $\sin_p$ functions have attracted a great deal of attention recently, especially in connection with the one-dimensional p-Laplacian and with the sharp estimation of the approximation numbers of embeddings. For example, the functions $\sin_p(n\pi_p x)$ turn out to be the eigenfunctions of the p-Laplacian eigenvalue problem

$$\left.\begin{aligned} &-(|u'|^{p-2}u')' = \lambda |u|^{p-2}u \ \text{ on } (0,1), \\ &u(0) = u(1) = 0, \end{aligned}\right\} \tag{1.4}$$

corresponding to eigenvalues $\lambda_n = (p-1)(n\pi_p)^p \qquad (n \in \mathbf{N})$. We refer to [2], [3] and [4] for further information and additional references on these functions and their applications. A fascinating account of early work on generalisations of trigonometric functions, is given in [5].

The only paper of which we are aware that deals with the basis properties of the $\sin_p$ functions is that of Binding et al [1], in which it is shown that if $12/11 \leq p < \infty$, they form a basis of $L_q(0,1)$ for all $q \in (1,\infty)$. The proof proceeds by constructing a homeomorphism of $L_q(0,1)$ onto itself that maps $\sin(n\pi x)$ onto $\sin_p(n\pi_p x) \ \ (n \in \mathbf{N})$, and while this method allows a slightly smaller value than $12/11$ to be obtained, it does not enable the basis property to be established for p arbitrarily close to 1. In contrast to this, we construct a basis of the dual of $L_q(0,1)$ that forms a biorthogonal system with the $\sin_p(n\pi_p x)$ functions. This enables us to conclude, by means of a general theorem about bases in Banach spaces, that the $\sin_p(n\pi_p x)$ functions form a basis of $L_q(0,1)$ for all $p, q \in (1,\infty)$.

2 Preliminaries and technical results

Throughout the paper p will stand for a number in $(1,\infty)$, $p' = p/(p-1)$, $I = [0,1]$ and $\|.\|_p$ will denote the usual norm on the Lebesgue space $L_p(I)$.

The function $\sin_p$ is defined on $[0, \pi_p/2]$ by (1.3): note that F_p is strictly increasing, as is $\sin_p$. Extension to $[0, \pi_p]$ is achieved by setting

$$\sin_p(t) = \sin_p(\pi_p - t) \quad \text{for} \;\; t \in [\pi_p/2, \pi_p];$$

further extension to $[-\pi_p, \pi_p]$ is made by oddness, and finally $\sin_p$ is extended to $\mathbf{R}$ by $2\pi_p$-periodicity. This extension is in $C^1(\mathbf{R})$. Note that $\sin_p(0) = 0$ and $\sin_p(\pi_p/2) = 1$. We define $\cos_p : \mathbf{R} \to \mathbf{R}$ by

$$\cos_p t = \frac{d}{dt} \sin_p t, \quad t \in \mathbf{R}; \tag{2.1}$$

$\cos_p$ is even, $2\pi_p$-periodic and odd about $\pi_p/2$. Moreover,

$$|\sin_p t|^p + |\cos_p t|^p = 1, \qquad t \in \mathbf{R}. \tag{2.2}$$

The number π_p is easily shown to be given by

$$\pi_p = 2p^{-1}\Gamma(1/p')\Gamma(1/p) = \frac{2\pi}{p \sin(\pi/p)}. \tag{2.3}$$

For shortness we shall write

$$e_n(t) = \sin(n\pi t), \; x_n(t) = \sin_p(n\pi_p t) \quad (n \in \mathbf{N}, \; t \in I). \tag{2.4}$$

We recall that a sequence $\{y_n\}_{n\in\mathbf{N}}$ of elements of a Banach space Y is called a (Schauder) basis if, for every $y \in Y$, there is a unique sequence $\{a_n\}_{n\in\mathbf{N}}$ of scalars such that

$$y = \sum_{n=1}^{\infty} a_n y_n, \tag{2.5}$$

the series converging in the norm of Y. It can be shown (see [7], Proposition II.B.6, p.37) that the partial sum projections $P_N : Y \to Y$ defined by

$$P_N\left(\sum_{n=1}^{\infty} a_n y_n\right) = \sum_{n=1}^{N} a_n y_n \quad (N \in \mathbf{N}) \tag{2.6}$$

have the property that

$$\sup_{N\in\mathbf{N}} \|P_N\| < \infty. \tag{2.7}$$

Since each x_n is continuous on I it is in every $L_q(I)$ $(1 < q < \infty)$; and as the e_k form a basis in $L_q(I)$, x_n has a Fourier sine expansion, converging in each $L_q(I)$:

$$x_n(t) = \sum_{k=1}^{\infty} \hat{x}_n(k) \sin(k\pi t), \tag{2.8}$$

where

$$\hat{x}_n(k) = 2\int_0^1 x_n(t) \sin(k\pi t) dt. \tag{2.9}$$

The symmetry of x_1 about $t = 1/2$ means that

$$\hat{x}_1(k) = 0 \quad \text{when } k \text{ is odd,} \tag{2.10}$$

and that

$$\begin{aligned}
\hat{x}_n(k) &= 2\int_0^1 x_1(nt)\sin(k\pi t)dt \\
&= 2\sum_{m=1}^{\infty} \hat{x}_1(m)\int_0^1 \sin(k\pi t)\sin(mn\pi t)dt \\
&= \begin{cases} \hat{x}_1(m), & \text{if } mn = k \text{ for some odd } m, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{2.11}$$

To estimate $\hat{x}_1(m)$, we follow [1] and note that since $x_1''(t) < 0$ for all $t \in (0, 1/2)$, integration by parts twice gives

$$\begin{aligned}
|\hat{x}_1(m)| &= 4\left|\int_0^{1/2} x_1(t)\sin(m\pi t)dt\right| \\
&= \left|\frac{-4}{(m\pi)^2}\int_0^{1/2} x_1''(t)\sin(m\pi t)dt\right| \\
&\le \frac{-4}{(m\pi)^2}\int_0^{1/2} x_1''(t)dt \\
&= \frac{4\pi_p}{(m\pi)^2}.
\end{aligned} \tag{2.12}$$

Now let A be the infinite matrix with $(i,j)^{th}$-entry $a_{i,j}$ $(i, j, \in \mathbf{N})$, where

$$a_{ij} = \begin{cases} \hat{x}_1(m), & \text{if } i = jm \ \text{ for some odd integer } m, \\ 0, & \text{otherwise.} \end{cases} \tag{2.13}$$

Evidently A is lower triangular, with each diagonal element equal to $\hat{x}_1(1)$. In view of the structure of A, there exists a matrix $B = (b_{ij})_{i,j \in \mathbf{N}}$ such that

$$BA = (\delta_{ij}), \tag{2.14}$$

the identity matrix; B is lower triangular, with each diagonal element equal to $1/\hat{x}_1(1)$. We use B to define elements of the dual $L_q(I)^*$ of $L_q(I)$ as follows. Let $f \in L_q(I)$, so that

$$f = \sum_{j=1}^{\infty} \hat{f}(j)e_j.$$

For each $i \in \mathbf{N}$, define a functional f_i^* on $L_q(I)$ by

$$f_i^*(f) = \sum_{j=1}^{i} b_{ij} \hat{f}(j). \tag{2.15}$$

We claim that $f_i^* \in L_q(I)^*$. As the linearity of f_i^* is clear, it remains to show that f_i^* is bounded. Since

$$|f_i^*(f)| \leq \max_{1 \leq j \leq i} |b_{ij}| \ \max_{j \in \mathbf{N}} |\hat{f}(j)|$$

and

$$|\hat{f}(j)| \leq 2\|f\|_1 \leq 2\|f\|_q \quad (j \in \mathbf{N}),$$

it follows that $f_i^* \in L_q(I)^*$.

Next we claim that the systems $\{x_i\}$ and $\{f_i^*\}$ are biorthogonal, by which we mean that

$$f_i^*(x_j) = \delta_{ij} \quad (i, j \in \mathbf{N}). \tag{2.16}$$

To verify this, note that

$$f_i^*(x_j) = \sum_{k=1}^{j} b_{ik} \hat{x}_j(k) = \sum_{k=1}^{\infty} b_{ij} a_{kj} = \delta_{ij}.$$

Since $f_i^* \in L_q(I)^*$, there exists $f_i \in L_{q'}(I)$ such that for all $f \in L_q(I)$,

$$f_i^*(f) = \int_0^1 f_i(t) f(t) dt.$$

In fact, f_i is given by

$$f_i = \sum_{j=1}^{i} b_{ij} e_j. \tag{2.17}$$

To check this, write

$$f = \sum_{k=1}^{\infty} \hat{f}(k) e_k$$

and observe that, with f_i given by (2.17),

$$\begin{aligned} \int_0^1 f_i(t) f(t) dt &= \int_0^1 \left(\sum_{j=1}^{i} b_{ij} e_j(t) \right) \left(\sum_{k=1}^{\infty} \hat{f}(k) e_k(t) \right) dt \\ &= \sum_{j=1}^{i} b_{ij} \hat{f}(j) = f_i^*(f). \end{aligned}$$

Finally, we note that the $L_q(I)$-norm of each x_i can be calculated. In fact, with

$$\begin{aligned} I &= \int_0^1 |\sin_p(i\pi_p x)|^q dx = i\int_0^{1/i} (\sin_p(i\pi_p x))^q \, dx \\ &= \int_0^1 (\sin_p(\pi_p t))^q \, dt = 2\int_0^{1/2} (\sin_p(\pi_p t))^q \, dt, \end{aligned}$$

the substitutions $u = \sin_p(\pi_p t)$ and then $u^p = w$ give

$$\begin{aligned} I &= \frac{2}{\pi_p}\int_0^1 u^q(1-u^p)^{-1/p}du = \frac{2}{p\pi_p}\int_0^1 u^{(q+1)/p-1}(1-w)^{-1/p}dw \\ &= \frac{2}{p\pi_p}B((q+1)/p, 1/p'), \end{aligned}$$

where B is the beta function. Hence for all $i \in \mathbf{N}$,

$$\|x_i\|_q = \left\{\frac{2}{p\pi_p}B((q+1)/p, 1/p')\right\}^{1/q}. \tag{2.18}$$

3 The main result

After the technical preparation of §2, we can now obtain the desired result fairly quickly. The strategy is to show that the f_i^* defined by (2.15) form a basis of $L_q(I)^*$, and then to use the biorthogonality of the systems $\{x_i\}$ and $\{f_i^*\}$ to conclude, via general theorems, that the x_i form a basis of $L_q(I)$.

Lemma 3.1 *For any $q \in (1,\infty)$, the sequence $\{f_i^*\}_{i\in\mathbf{N}}$ is complete in $L_q(I)^*$ in the sense that its closed linear span is $L_q(I)^*$.*

Proof. For each $n \in \mathbf{N}$ set $P_n = \mathrm{sp}\{f_1^*, f_2^*, ..., f_n^*\}$, the span of $f_1^*, f_2^*, ..., f_n^*$. Then $s^* \in P_n$ if and only if there exist $d_1, ..., d_n \in \mathbf{R}$ such that for all $f = \sum_{j=1}^{\infty} \hat{f}(j)e_j \in L_q(I)$,

$$s^*(f) = \sum_{j=1}^{n} d_j \hat{f}(j). \tag{3.1}$$

Moreover,

$$s^*(f) = \int_0^1 s(t)f(t)dt, \tag{3.2}$$

where

$$s = \sum_{j=1}^{n} d_j e_j. \tag{3.3}$$

Now let $g^* \in L_q(I)^*$ and let $g \in L_{q'}(I)$ be such that

$$g^*(f) = \int_0^1 g(t)f(t)dt, \quad f \in L_q(I).$$

Since $g \in L_{q'}(I)$, the basis property of the e_i in $L_{q'}(I)$ means that

$$g = \sum_{i=1}^{\infty} \hat{g}(i)e_i,$$

and for each $N \in \mathbf{N}$,

$$\|g - \sum_{i=1}^{N} \hat{g}(i)e_i\|_{q'} = \|\sum_{i=N+1}^{\infty} \hat{g}(i)e_i\|_{q'} \to 0 \text{ as } N \to \infty. \tag{3.4}$$

For each $n \in \mathbf{N}$ put

$$g_n = \sum_{i=1}^{n} \hat{g}(i)e_i \quad (\in L_{q'}(I))$$

and

$$g_n^*(f) = \int_0^1 g_n(t)f(t)dt, \quad f \in L_q(I).$$

From (3.2) we see that $g_n^* \in P_n$. For every $f \in L_q(I)$ we have, with the help of Hölder's inequality,

$$\begin{aligned} |g^*(f) - g_n^*(f)| &= \left| \int_0^1 (g(t) - g_n(t))f(t)dt \right| \\ &\leq \|g - g_n\|_{q'} \|f\|_q. \end{aligned}$$

Hence by (3.4),

$$\sup_{\|f\|_q \leq 1} |g^*(f) - g_n^*(f)| \leq \|g - g_n\|_{q'} \to 0$$

as $n \to \infty$, so that $g_n^* \to g^*$ in $L_q(I)^*$, as required. ■

Lemma 3.2 *Let $q \in (1, \infty)$. There is a sequence $\{u_n^*\}$ of bounded linear maps of $L_q(I)^*$ to itself (i.e. endomorphisms of $L_q(I)^*$) such that*
(i) $u_n^*(x^*) = x^*$ *for all* $x^* \in P_n$ $(n \in \mathbf{N})$,
(ii) $u_n^*(x^*) = 0$ *for all* $x^* \in P^{(n)}$ $(n \in \mathbf{N})$
and
(iii) $1 \leq C := \sup_{n \in \mathbf{N}} \|u_n^*\| < \infty$.

Here $P_n = \text{sp}\{f_1^*, ..., f_n^*\}$, $P^{(n)} = \text{sp}\{f_{n+1}^*, f_{n+2}^*, ...\}$ *(the closed linear span of* $f_{n+1}^*, f_{n+2}^*, ...$*) and the* f_i^* *are as defined in (2.15).*

Proof. Let $s^* \in P_n$ and let $s = \sum_{i=1}^{n} d_i e_i \in L_{q'}(I)$ be such that

$$s^*(f) = \int_0^1 s(t)f(t)dt, \quad f \in L_q(I).$$

Given $r^* \in P^{(n)}$, there exists $r = \sum_{i=n+1}^{\infty} c_i e_i \in L_{q'}(I)$ such that

$$r^*(f) = \int_0^1 r(t)f(t)dt, \quad f \in L_q(I).$$

For each $n \in \mathbf{N}$, define u_n on $L_{q'}(I)$ by

$$u_n(x) = \sum_{i=1}^{n} \hat{x}(i)e_i, \quad x = \sum_{i=1}^{\infty} \hat{x}(i)e_i \in L_{q'}(I).$$

We claim that each u_n is an endomorphism of $L_{q'}(I)$ and that we may define an endomorphism u_n^* of $L_q(I)^*$ by

$$u_n^*(x^*)(f) = \int_0^1 u_n(x)(t)f(t)dt, \quad f \in L_q(I), x^* \in L_q(I)^*,$$

where $x \in L_{q'}(I)$ is such that

$$x^*(f) = \int_0^1 x(t)f(t)dt, \quad f \in L_q(I).$$

Plainly u_n is a linear map from $L_{q'}(I)$; and it is bounded, for as the e_i form a basis of $L_{q'}(I)$, we see from (2.7) that there is a constant C such that for all $n \in \mathbf{N}$,

$$\|u_n(x)\|_{q'} \le C\|x\|_{q'}, \quad x \in L_{q'}(I). \tag{3.5}$$

Moreover, for every $f = \sum_{i=1}^{n} c_i e_i$, $g = \sum_{i=n+1}^{\infty} c_i e_i \in L_{q'}(I)$ we have

$$u_n(f) = f, \quad u_n(g) = 0.$$

Next we justify the claim that each u_n^* is an endomorphism on $L_q(I)^*$. Linearity is obvious; and for each $x^* \in L_q(I)^* \setminus \{0\}$,

$$\begin{aligned} \|u_n^*(x^*)\|_{L_q(I)^*}/\|x^*\|_{L_q(I)^*} &= \left\{ \sup_{\|f\|_q \le 1} |u_n^*(x^*)(f)| \right\} / \sup_{\|f\|_q \le 1} |x^*(f)| \\ &= \|u_n(x)\|_{q'}/\|x\|_{q'} \le C \end{aligned} \tag{3.6}$$

the inequality following from (3.5). Hence

$$\|u_n^*\| \le C \quad (n \in \mathbf{N}).$$

To establish a lower bound for the $\|u_n^*\|$, let $x = e_1$. Then by (3.6),

$$\|u_n^*(x^*)\|_{L_q(I)^*}/\|x^*\|_{L_q(I)^*} = \|u_n(x)\|_{q'}/\|x\|_{q'} = 1.$$

The proof is complete. ∎

Theorem 3.3 *For each $q \in (1, \infty)$, the system $\{f_i^*\}_{i \in \mathbf{N}}$ defined by (2.15) is a basis of $L_q(I)^*$.*

Proof. By Lemma 3.1, $\{f_i^*\}_{i \in \mathbf{N}}$ is complete in $L_q(I)^*$. Now apply Theorem 7.1 of [6], which shows that such a system $\{f_i^*\}$ is a basis if there is a sequence of endomorphisms of $L_q(I)^*$ having the properties ensured by Lemma 3.2. ■

The main result of the paper is now virtually immediate.

Theorem 3.4 *The functions x_i ($i \in \mathbf{N}$) form a basis in $L_q(I)$ for every $q \in (1, \infty)$.*

Proof. By (2.16), $\{x_i\}$ and $\{f_j^*\}$ are biorthogonal; by Theorem 3.3, $\{f_j^*\}$ is a basis of $L_q(I)^*$. Corollary 12.1 of [6] tells us that under these conditions, $\{x_i\}$ is a basis of $L_q(I)$. ■

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