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General Information:

This is a midterm from a previous semester. This means:

- This midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to this midterm.
- This midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take this exam:

You should treat this midterm should be as the actual exam. This means:

- "Practice like you play." Schedule 55 uninterrupted minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and projects with this intent, and there will be a problem on your midterm that will require you to apply the material in an unfamiliar setting.

How to use the solutions:

DO NOT JUST READ THE SOLUTIONS!!!

The least important aspect of the solutions is learning the steps necessary to solve a specific problem. You should be looking for the concepts required to provide solutions. Content may not be recycled, but concepts will be!

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

- -

Math 1152 Name: Solutions

Midterm 1 OSU Username (name.nn): ______

Autumn 2016 Lecturer: ______

Recitation Instructor: ______

Form A Recitation Time: ______

Instructions

- You have **55 minutes** to complete this exam. It consists of 6 problems on 12 pages including this cover sheet. Page 11 has some potentially useful formulas and both Pages 11 and 12 may be used extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	12	
2	20	
3	20	
4	14	
5	16	
6	18	
Total	100	

1. **Multiple Choice** [12 pts]

Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

I. The density for a wire from $x = 0$ to $x = 6$ is given by:

$$\rho(x) = \begin{cases} 4 & , \quad 0 \le x \le 2 \\ 6 - x & , \quad 2 < x \le 6 \end{cases}$$

What is the mass of the segment of the wire from $x = 0$ to $x = 4$?

A. 4 units $\boxed{\text{B. 14 units}}$ C. 16 units D. None of the above

$$m = \int_0^4 \rho(x)\,dx = \int_0^2 4\,dx + \int_2^4 (6-x)\,dx$$

$$= 4x\Big|_0^2 + \left[6x - \tfrac{1}{2}x^2\right]_2^4 = (8-0) + (16-10) = \boxed{14}$$

II. Find a function $f(x)$ that satisfies $\int f(x)\,dx = 2x\cos(x^2) + C$.

A. $f(x) = -2\sin(x^2)$

$\boxed{\text{B. } f(x) = 2\cos(x^2) - 4x^2\sin(x^2)}$

C. $f(x) = x^2\sin(x^2)$

D. $f(x) = \sin(x^2) + C$

E. No such function exists

F. None of the above

If $\int f(x)\,dx = 2x\cos(x^2) + C$, then $f(x) = \frac{d}{dx}\left[2x\cos(x^2) + C\right]$

$$= 2\cos(x^2) - 4x^2\sin(x^2).$$

III. A spring obeys a modified Hooke's Law, for which the force required to stretch the spring x meters from equilibrium is given by:

$$F(x) = kx^2.$$

Suppose $80\,J$ of work is required to stretch a spring $2\,m$ from its equilibrium position at $x = 0$. What is the value of the spring constant k (in N/m^2)?

A. $k = 20$ $\boxed{\text{B. } k = 30}$ C. $k = 40$ D. $k = 50$ E. None of the above

$$W = \int_0^2 F(x)\,dx$$

$$80 = \int_0^2 kx^2\,dx$$

$$80 = \tfrac{1}{3}kx^3\Big|_0^2$$

$$80 = \tfrac{1}{3}k(8) - 0 \rightarrow \boxed{k = 30}$$

2. **Short Answer** [20 pts]

Answer the following questions and provide as much justification as is requested.

I. [6 pts] A student claims that:

$$\int \tan^2 x\,dx = \frac{1}{3}\tan^3 x + C.$$

Determine if this student is correct or incorrect. If the student is incorrect, justify your response by explaining why the proposed solution cannot be the correct antiderivative.

If the student is correct, then $\frac{d}{dx}\left[\frac{1}{3}\tan^3 x + C\right]$ would be $\tan^2 x$.

But, $\frac{d}{dx}\left[\frac{1}{3}\tan^3 x + C\right] = \tan^2 x \cdot \frac{d}{dx}(\tan x)$ by chain rule

$= \tan^2 x \cdot \sec^2 x.$

$\neq \tan^2 x$

II. [6 pts] Suppose that the region R is bounded by $y = 4 - 2x$, $y = 0$, and $x = 0$. CIRCLE the correct response to each question. NO justification is necessary!

A. [3 pts] If you revolve R about the $x-$axis and integrate with respect to y, which method should be used to find the volume of the solid?

↳ horizontal rect.

DISK METHOD $\qquad$ $\boxed{\text{SHELL METHOD}}$

B. [3 pts] The Washer Method can be used to write the resulting volume as an integral with respect to x if you revolve R about the line:

rect are ∥ axis of rot ⇒ Shell.

- Washers → rect are ⊥ axis of rot.
- Int wrt x → vert rect

⇒ axis must be horizontal.

$x = 5$ $\qquad$ $\boxed{y = -2}$

III. [8 pts] Calculate the following antiderivatives. A perfect score is worth 8 points. You will lose 2 points for each incorrect response, but you cannot score below a 0. Thus, the possible scores for this problem are 0, 2, 4, 6, or 8 points.

A. $\int \frac{1}{e^x}\,dx =$ $-e^{-x} + C$

$\int e^{-x}\,dx$

B. $\int \sin(2x)\,dx =$ $-\frac{1}{2}\cos 2x + C$

C. $\int (\sqrt{x} + 2)^2\,dx =$ $\frac{1}{2}x^2 + \frac{8}{3}x^{3/2} + 4x + C$

$= \int (x^{1/2} + 2)(x^{1/2} + 2)\,dx$

$= \int (x + 4x^{1/2} + 4)\,dx$

D. $\int \frac{2x + 1}{2x}\,dx =$ $x + \frac{1}{2}\ln|x| + C$

$= \int \left(\frac{2x}{2x} + \frac{1}{2x}\right)dx$

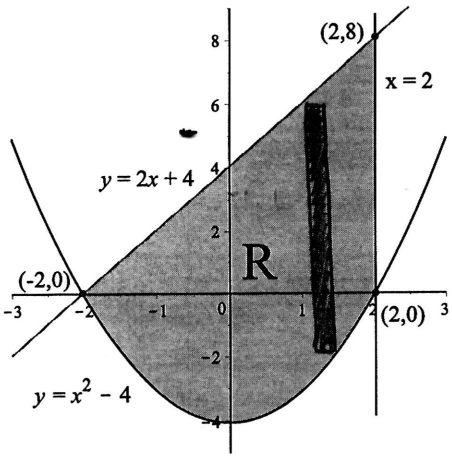
$= \int \left(1 + \frac{1}{2}\cdot\frac{1}{x}\right)dx$

* Many of these came from calculator or Quiz 1!

3. [20 pts] The region R is bounded by the curves $y = 2x + 4$, $y = x^2 - 4$ and $x = 2$.

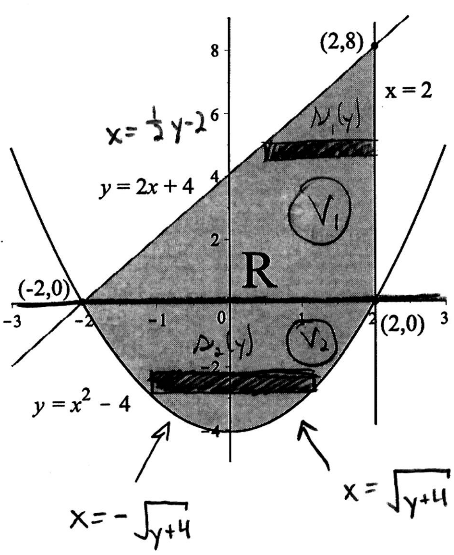
I. **Set up**, but **do not evaluate**, an integral or a sum of integrals that would give the area of R if evaluated.

$$\boxed{A = \int_{x=-2}^{x=2} \left[(2x-4) - (x^2-4)\right] dx}$$



Easier to use vertical rect than horizontal ones!

II. The base of a solid is the region R. Cross-sections through the solid perpendicular to the **y-axis** are squares. **Set up**, but **do not evaluate**, an integral or sum of integrals that would give the volume of this solid. The area of a square is $A(y) = [s(y)]^2$:



$$V_1 = \int_{y=0}^{y=8} A(y)\, dy$$

$$= \int_{y=0}^{y=8} \left[2 - \left(\tfrac{1}{2}y - 2\right)\right]^2 dy$$

$$V_1 = \int_{y=0}^{y=8} \left(4 - \tfrac{1}{2}y\right)^2 dy$$

$$V_2 = \int_{y=-4}^{y=0} \left[\sqrt{y+4} - \left(-\sqrt{y+4}\right)\right]^2 dy$$

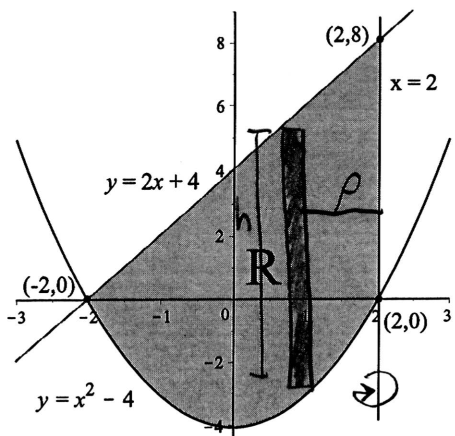
$$= \int_{y=-4}^{y=0} \left[2\sqrt{y+4}\right]^2 dy$$

$$= \int_{y=-4}^{y=0} 4(y+4)\, dy$$

Thus, $\boxed{V = \int_{y=-4}^{y=0} (4y+16)\,dy + \int_{y=0}^{y=8} \left(4 - \tfrac{1}{2}y\right)^2 dy}$

(*Problem 3 continued*)

III. A solid of revolution is formed by revolving the region R on the previous page about the line $x = 2$. **Set up**, but **do not evaluate**, an integral or sum of integrals **with respect to x** that would give the volume of this solid. You do not need to simplify your final answer.



- We want to use vertical rectangle (to ensure we only need a single integral)
  → integrate wrt x.
- The rect are ∥ axis of rotation
  → Use shells.

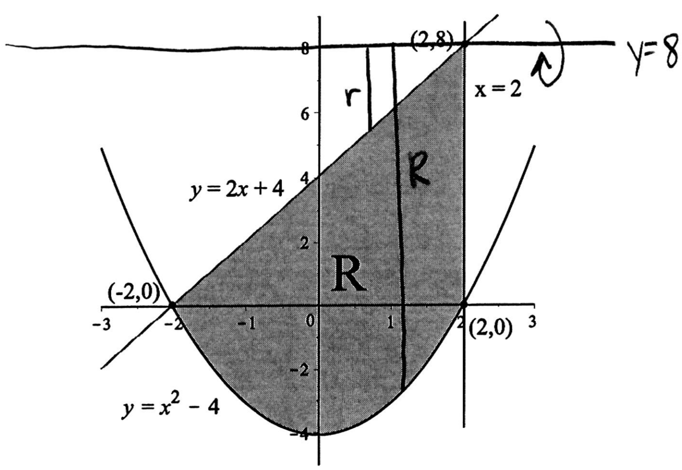
$$V = \int_{x=-2}^{x=2} 2\pi \rho h \, dx$$

$$\boxed{V = \int_{x=-2}^{x=2} 2\pi (2-x)(-x^2+2x+8)\, dx}$$

ρ: $2-x$

$h = (2x+4) - (x^2-4)$

IV. A solid of revolution is formed by revolving the region R on the previous page about the line $y = 8$. **Set up**, but **do not evaluate**, an integral or sum of integrals that would give the volume of this solid using the **Washer Method**.



- Washers → rect are ⊥ axis of rot
- axis of rot is horizontal, so rect must be vertical
  ⇒ int wrt x

$$V = \int_{x=-2}^{x=2} \pi (R^2 - r^2)\, dx$$

$$\boxed{V = \int_{x=-2}^{x=2} \pi \left[(12-x^2)^2 - (4-2x)^2 \right] dx}$$

R: dist from axis of rot to outer curve

$= 8 - (x^2 - 4)$

r: dist from axis to inner curve

$= 8 - (2x+4)$

4. [14 pts] Find the length of the curve $y = \frac{2}{3}(1+x^2)^{3/2} + 2$ from $x = 0$ to $x = 3$.

- $y' = (1+x^2)^{1/2} \cdot 2x$ ← Chain rule.
- $(y')^2 = \left[(1+x^2)^{1/2} \cdot 2x\right]^2$
  $= (1+x^2)\, 4x^2$
  $= 4x^2 + 4x^4.$
- $1+(y')^2 = 1 + 4x^2 + 4x^4$
  $= (1+2x)^2$ ← perfect square!

Now, $\ell = \int_0^3 \sqrt{1+(y')^2}\, dx$

$= \int_0^3 \sqrt{(1+2x)^2}\, dx$

$= \int_0^3 |1+2x|\, dx$ ← $1+2x > 0$ on $[0,3]$, so $|1+2x| = 1+2x$ there.

$= \int_0^3 (1+2x)\, dx$

$= \left[x + x^2\right|_0^3$

$= (3+9) - 0$

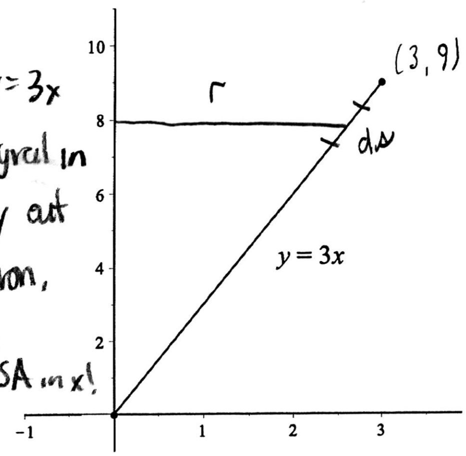
$= \boxed{12}$

5. [16 pts] (Surface Area and Work)

A hollow tank is obtained by revolving the portion of the line $y = 3x$ from $x = 0$ to $x = 3$ about the y-axis. $x = \frac{1}{3}y$

I. **Set up**, but **do not evaluate**, an integral that if evaluated would give the surface area of the tank.

*If you let $y=3x$ in the SA integral in y and carry out the substitution, you'll get the integral for SA in x!



Way 1: let $ds = \sqrt{1+(y')^2}\,dx$

$= \sqrt{1+(3)^2}\,dx$

$= \sqrt{10}\,dx.$

We need r as a function of x. Since $r = x$ by the picture,

$SA = \int 2\pi r\,ds$

$$\boxed{SA = \int_{x=0}^{x=3} 2\pi x\sqrt{10}\,dx}$$

Way 2: $ds = \sqrt{1+(x')^2}\,dy$

$= \sqrt{1+(\frac{1}{3})^2}\,dy$

$= \sqrt{\frac{10}{9}}\,dy$

$= \frac{1}{3}\sqrt{10}\,dy.$

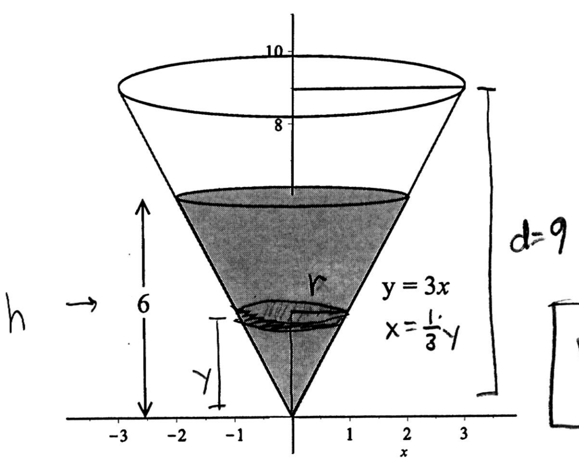
We need r as a function of y. From the picture, $r = x = \frac{1}{3}y$, so:

$SA = \int 2\pi r\,ds$

$$= \boxed{\int_{y=0}^{y=9} 2\pi \cdot \frac{1}{3}y \cdot \frac{1}{3}\sqrt{10}\,dy}$$

II. The tank is now filled to a height of 6 m with Liquid X, whose density is 900 kg/m^3, as shown below.

Set up, but **do not evaluate**, an integral or sum of integrals, that would give the work required to pump the liquid to the top of the tank. Use $g = 9.8\,m/s^2$.



$$W = \int_{y=0}^{y=h} \rho g\, A(y)(d-y)\,dy$$

$$= \int_0^6 900(9.8)\,\frac{\pi}{9}\,y^2(9-y)\,dy$$

$$\boxed{W = \int_0^6 980\pi\,(9y^2 - y^3)\,dy}$$

Since cross-sections are circles,

$A(y) = \pi[r(y)]^2$

Since $r = x = \frac{1}{3}y$

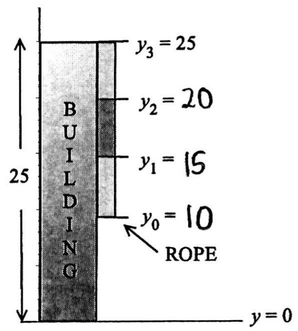
$A(y) = \frac{\pi}{9}y^2$

6. [18 pts] **Read the following problem carefully!**

The amount of work, W, required to move a particle of mass m a vertical distance d above its starting height is given by the formula:

$$W = mgd.$$

A heavy rope is 15 m long and has a constant (linear) density of $\rho = 2\,kg/m$. The rope hangs over the edge of a 25 m building, as shown below:



The work required to pull the rope to the top of the building cannot be computed using the above formula because the distance each part of the rope must be moved is different!

I. (The Approximate Work)

A. On the figure above, the rope has been divided into three segments of equal height, Δy. Calculate Δy.

$\Delta y =$ 5 m.

The rope is 15 m long, so each of the 3 segments is 5 m long.

B. Find the mass, m_2, of the *middle* segment of the rope.

$m_2 =$ 10 kg

The density is const, so $m = \rho\Delta y = 2(5) = 10$.

C. The y values of the *bottom* of each segment of rope are indicated above. **Find** the values y_0, y_1, and y_2 and **label** them on the figure (No justification is necessary).

Make sure you READ this! →

By approximating that the *middle* segment is a particle located at y_2:

D. **Find** the distance d_2 the *middle* segment must be moved to reach the top of the building.

$d_2 =$ 5m

The rope is at $y_2 = 20$, so it must be moved up 5 m.

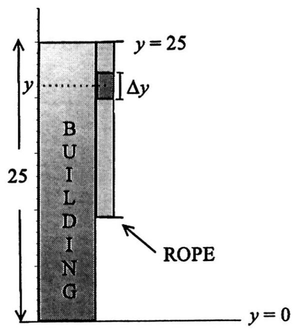
E. **Find** the approximate the work, W_2 required to move the *middle* segment to the top of the building by using the given formula. For computational convenience, take $g = 10\,m/s^2$.

$W_2 =$ 500 J

$W_2 = m_2 g d_2 = 10(10)(5) = 500\text{ J}$

II. (The Exact Work)

To compute the *exact* work required to pull the whole rope to the top of the building, slice the rope into many segments of equal height, Δy. One such segment, centered at a height y, is shown on the figure below:



A. Find the mass, Δm, of the indicated segment of the rope. Leave your answer in terms of Δy.

$\Delta m =$ $2\Delta y.$ $\quad \rho = \frac{\Delta m}{\Delta y} \rightarrow 2 = \frac{\Delta m}{\Delta y}$

By approximating that the segment is a particle located at height y:

B. **Find** the distance $d = d(y)$[1] that the segment must be moved to reach the top of the building in terms of y.

$d(y) =$ $25-y$

The particle is at y and must be moved to 25, so $d = 25-y$

C. **Find** the approximate the work, ΔW required to move the segment to the top of the building by using the formula given in the beginning of the problem. For computational convenience, take $g = 10\,m/s^2$ and leave your final answer in terms of y and Δy.

$\Delta W =$ $20(25-y)\Delta y$

$\Delta W = \Delta m \, g \, d(y)$ since we appx the segment as a particle.

$= 2\Delta y (10)(25-y)$

D. **Set up**, but **do not evaluate** an integral that would give the work required to pull the entire rope up to the top of the building.

$W = \int_{y=c}^{y=d} dW$, where $y=c$ is the location of the bottom segment $y=d$ is the location of the top segment.

$$\boxed{W = \int_{10}^{25} 20(25-y)\,dy}$$

[1]This is just notation that the distance will depend on the height y where the segment is found.