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General Information:

This is a midterm from a previous semester. This means:

- This midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to this midterm.
- This midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take this exam:

You should treat this midterm should be as the actual exam. This means:

- "Practice like you play." Schedule 55 uninterrupted minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and projects with this intent, and there will be a problem on your midterm that will require you to apply the material in an unfamiliar setting.

How to use the solutions:

DO NOT JUST READ THE SOLUTIONS!!!

The least important aspect of the solutions is learning the steps necessary to solve a specific problem. You should be looking for the concepts required to provide solutions. Content may not be recycled, but concepts will be!

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

- -

Math 1152 Name: Solutions

Midterm 2 OSU Username (name.nn): ______________________

Autumn 2016 Lecturer: ______________________

Recitation Instructor: ______________________

Form A Recitation Time: ______________________

Instructions

- You have **55 minutes** to complete this exam. It consists of 6 problems on 12 pages including this cover sheet. Page 11 has possibly helpful formulas and pages 11 and 12 may also be used for extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	12	
2	10	
3	26	
4	16	
5	16	
6	20	
Total	100	

1. **Multiple Choice** [12 pts]

Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

I. Suppose that $\{a_k\}$ is a sequence for which $a_k > 0$ for all $k \geq 1$ and

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{1}{2}.$$

Then, the Ratio Test guarantees:

Ratio test guarantees the series converges if $\lim_{k\to\infty} \frac{|a_{k+1}|}{a_k} < 1$ but doesn't tell us the value to which the series converges.

A. $\sum_{k=1}^{\infty} a_k$ diverges.

B. $\sum_{k=1}^{\infty} a_k$ converges to $\frac{1}{2}$.

$\boxed{\text{C. } \sum_{k=1}^{\infty} a_k \text{ converges but there is not enough information to determine its value.}}$

D. Nothing; the Ratio Test may not apply since we do not know a formula for a_k.

II. The series $\sum_{k=1}^{\infty} 3\left(\frac{1}{2}\right)^k$:

A. converges to 6.

$\boxed{\text{B. converges to 3.}}$

C. converges to 2.

D. diverges.

E. None of the above.

This is a geometric series with $r = \frac{1}{2} < 1$, so it converges.

$$\underbrace{\sum_{k=1}^{\infty} 3\left(\tfrac{1}{2}\right)^k}_{\text{can't use formula since } k \neq 0!} = \sum_{k=0}^{\infty} 3\left(\tfrac{1}{2}\right)^k - 3\left(\tfrac{1}{2}\right)^0 = \frac{3}{1-\frac{1}{2}} - 3 = \frac{3}{1/2} - 3 = \boxed{3}$$

III. Suppose that $\{a_k\}$ is a sequence for which $a_k \geq 0$ for all $k \geq 1$ and it is known that $\sum_{k=0}^{\infty} 2^k a_k$ converges. Then:

$\boxed{\text{A. } \sum_{k=0}^{\infty} a_k \text{ must converge.}}$

B. $\sum_{k=0}^{\infty} a_k$ must diverge.

C. $\sum_{k=0}^{\infty} a_k$ could converge or diverge.

D. None of the above.

Since $a_k > 0$, we can use the comparison test;
Since $a_k < 2^k a_k$ and $\sum 2^k a_k$ converges,
$\sum a_k$ must converge by the comparison test.

2. **Multiselect** [10 pts] Circle *all* of the responses that MUST be true for the problem below. Note that there may be more than one correct response or even no correct responses!

A perfect answer for this question is worth 10 points. You will be penalized 2 points for:

- each incorrect choice that you circle
- each correct choice that you do not circle.

Thus, the possible grades on this problem are 0, 2, 4, 6, 8 or 10. You cannot score below a 0 for this problem.

Problem: Suppose that $a_n > 0$ for all $n \geq 1$. Let $s_n = \sum_{k=1}^{n} a_k$ and suppose $\lim_{n\to\infty} s_n = 2$.

CIRCLE ALL of the statements below that MUST be TRUE.

A. $\sum_{k=1}^{\infty} a_k = 2$	B. $\sum_{n=1}^{\infty} a_n = 2$	C. $\{s_n\}$ MUST be monotonic.
D. $\lim_{n\to\infty} a_n = 0$	E. $\sum_{k=1}^{\infty} (a_k - 2) = 0$	F. $\{s_n\}$ MUST be bounded.
G. $\sum_{k=1}^{\infty} a_{k+1}$ converges.	H. $\sum_{k=1}^{\infty} a_k$ is geometric.	I. $\sum_{k=1}^{\infty} s_k$ MUST diverge.

A. True; By definition, $\sum_{k=1}^{\infty} a_k = L$ if and only if $\lim_{n\to\infty} s_n = L$.

B. True; This is the same question as A; both series represent the infinite series $a_1 + a_2 + a_3 + \dots$

C. True; Since $a_n > 0$ for all n, $s_{n+1} = s_n + a_{n+1} > s_n$ for all n, so s_n is increasing and thus monotonic.

D. True; A corollary to the divergence theorem assures if $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n\to\infty} a_n = 0$.

E. False; $\lim_{k\to\infty} a_k = 0$ by D, so $\lim_{k\to\infty} (a_k - 2) = -2 \neq 0$. $\sum_{k=1}^{\infty} (a_k - 2)$ diverges by the divergence theorem.

F. True; $\lim_{n\to\infty} s_n$ exists so $\{s_n\}$ must be bounded.

G. True; Reindexing, $\sum_{k=1}^{\infty} a_{k+1} = \sum_{k=2}^{\infty} a_k$, which converges since $\sum_{k=1}^{\infty} a_k$ converges.

H. False; No information is provided about a_k.

I. True; $\lim_{n\to\infty} s_n = 2 \neq 0$, so $\sum s_n$ diverges by divergence test.

3. **Short Answer** [26 pts]

Answer each of the following questions below and provide as much justification as requested.

I. Suppose that $\{a_n\}_{n\geq 1}$ is a sequence for which $s_n = \dfrac{n^3+3^n}{3^n}$, where as usual $s_n = \sum_{k=1}^{n} a_k$ for $n \geq 1$. Find the following. Clearly **explain** your responses!

A. [4 pts] $a_1 + a_2 + a_3 = \underline{\quad 2 \quad}$.

By definition $a_1 + a_2 + a_3 = s_3 = \dfrac{(3)^3 + 3^{(3)}}{3^{(3)}} = \boxed{2.}$

B. [6 pts] Determine whether $\sum_{k=4}^{\infty} a_k$ converges or diverges. If it converges, give the value to which it converges.

- $\sum_{k=1}^{\infty} a_k$ converges if and only if $\lim_{n\to\infty} s_n$ exists. Note:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\underset{\substack{\uparrow \\ =0 \text{ by growth rates}}}{\frac{n^3}{3^n}} + \frac{\cancel{3^n}}{\cancel{3^n}}\right) = \underline{\underline{1}}.$$

Hence $\sum_{k=1}^{\infty} a_k$ converges to 1.

- Since $\sum_{k=4}^{\infty} a_k = \underbrace{\sum_{k=1}^{\infty} a_k}_{1} - \underbrace{(a_1+a_2+a_3)}_{2}$, we have $\boxed{\sum_{k=4} a_k \text{ converges to } -1}$

C. [4 pts] Determine whether $\sum_{k=1}^{\infty} s_k$ converges or diverges. If it converges, give the value to which it converges.

$\lim_{k\to\infty} s_k = 1$ by above. Since $\lim_{k\to\infty} s_k \neq 0$,

$\boxed{\sum_{k=1}^{\infty} s_k \text{ diverges by the divergence test}}$

(Short Answer continued)

II. [4 pts] Give the general partial fraction decomposition for $\frac{4x+7}{x^4+x^2}$. DO NOT SOLVE FOR THE CONSTANTS!

$$\frac{4x+7}{x^4+x^2} = \frac{4x+7}{\underbrace{x^2}_{\text{repeated linear}}\underbrace{(x^2+1)}_{\text{irreducible quadratic}}} = \boxed{\frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{x^2+1}}$$

III. [4 pts] Write the expression below in summation notation:

$$1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots + \left(\frac{2}{3}\right)^{17}$$

$$\boxed{\sum_{k=0}^{17} \left(\frac{2}{3}\right)^k}$$

IV. [4 pts] For which real values of a is the integral $\int_0^1 \frac{1}{x-2a}\,dx$ improper? Explain your answer!

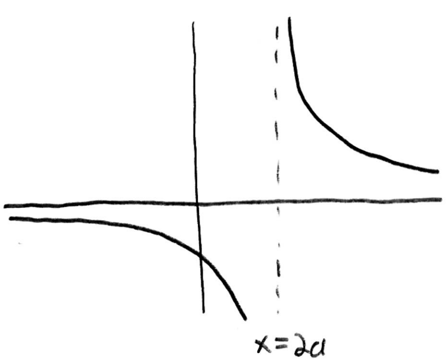
The integral is improper if $\frac{1}{x-2a}$ is unbounded on $[0,1]$.

The integrand is unbounded when $x-2a=0$

$x=2a$.

The integral is improper if this x-value occurs in the interval of integration

$\Rightarrow$ $\boxed{\text{The integral is improper if } 0 \leq 2a \leq 1 \quad 0 \leq a \leq \frac{1}{2}}$



The graph of $y = \frac{1}{x-2a}$ has an asymptote at $x=2a$. If this x-value is in the interval of integration the integral will be improper.

4. [16 pts] (Trigonometric substitution and improper integrals)

I. [11 pts] Use an appropriate trigonometric substitution to show that for $x > 0$:

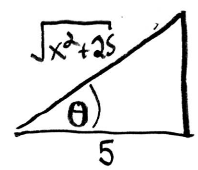
$$\int \frac{50}{(x^2+25)^{3/2}}\, d\cancel{x} = \frac{2x}{\sqrt{x^2+25}} + C$$

- x^2+25 is of the form u^2+a^2, $u=x$, $a=5$.

$\rightarrow$ use $u = a\tan\theta$.

- $x = 5\tan\theta$

$dx = 5\sec^2\theta$.



$x = 5\tan\theta$

$\tan\theta = \frac{x}{5}$

- Thus, $\int \frac{50}{(x^2+25)^{3/2}}\, dx = \int \frac{50}{(25\tan^2\theta+25)^{3/2}} \cdot 5\sec^2\theta\, d\theta$

$$= \int \frac{250}{(25\sec^2\theta)^{3/2}} \sec^2\theta\, d\theta$$

$$= \int \frac{250}{125\sec^3\theta} \sec^2\theta\, d\theta$$

$$= 2\int \frac{1}{\sec\theta}\, d\theta$$

$$= 2\int \cos\theta\, d\theta$$

$$= 2\sin\theta + C$$

$$= 2\cdot\frac{x}{\sqrt{x^2+25}} + C$$

$$= \boxed{\frac{2x}{\sqrt{x^2+25}} + C}$$

II. [5 pts] Determine whether the improper integral:

$$\int_0^\infty \frac{50}{(x^2+25)^{3/2}}\, dx$$

converges or diverges. If it converges, give the value to which it converges. Make sure you use proper notation in your solution!

The improper integral converges if and only if $\lim_{b\to\infty} \int_0^b \frac{50}{(x^2+25)^{3/2}}\, dx$ exists.

$$\lim_{b\to\infty} \int_0^b \frac{50}{(x^2+25)^{3/2}}\, dx = \lim_{b\to\infty} \left[\frac{2x}{\sqrt{x^2+25}}\right]_0^b \quad \text{by I.}$$

$$= \lim_{b\to\infty} \left[\frac{2b}{\sqrt{b^2+25}} - 0\right]$$

$$\frac{2b}{\sqrt{b^2+25}} = \frac{2b}{\sqrt{b^2}\sqrt{1+\frac{25}{b^2}}} = \frac{2\cancel{b}}{\cancel{b}\sqrt{1+\frac{25}{b^2}}}, \; b>0$$

$$= 2$$

$\boxed{\text{The improper integral converges to } 2.}$

5. [16 pts] Compute the following antiderivatives.

I. $\int \frac{4}{1-4x^2}\,dx.$

Since $\frac{4}{1-4x^2} = \frac{4}{(1-2x)(1+2x)}$, we can use partial fractions.

$$\frac{4}{(1-2x)(1+2x)} = \frac{A}{1-2x} + \frac{B}{1+2x}$$

$$4 = A(1+2x) + B(1-2x)$$ ← This holds for all x, so try convenient x-values first

$\underline{x = -\frac{1}{2}}$: $4 = 2B \rightarrow \underline{B = 2}$

$\underline{x = \frac{1}{2}}$: $4 = 2A \rightarrow \underline{A = 2}$

Thus, $\int \frac{4}{1-4x^2}\,dx = \int \left(\frac{2}{1-2x} + \frac{2}{1+2x}\right)dx$

$$= \boxed{-\ln|1-2x| + \ln|1+2x| + C}$$

Make sure you understand why the coefficients are NOT 2!

II. $\int x^2 \ln x\,dx.$

$\underline{\text{Use integration by parts}}$: since we can't easily integrate $\ln x$, let $u = \ln x$:

$u = \ln x$ $\qquad$ $dv = x^2\,dx$

$du = \frac{1}{x}\,dx$ $\qquad$ $v = \frac{1}{3}x^3$.

So: $\int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\,dx$

$$= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2\,dx$$

$$= \boxed{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$$

6. [20 pts] Determine whether each series below converges or diverges. If a series converges, determine if the series converges absolutely or conditionally.

In order to get full credit, you must *fully* justify your work by:

- **Stating** any convergence test you use and why the test applies.
- **Explaining** the conclusions of the test! This includes justifying any limits you need to compute!

You may quote any results about p-series or geometric series, but you must *clearly* explain why the type of series you are considering converges or diverges!

I. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k + k^6}{3^k}$

There are several ways to solve this; a few are listed below.

Way 1: Use Limit Comparison Test

Since $\frac{2^k+k^6}{3^k} > 0$, we can use LCT with $\sum \frac{2^k}{3^k}$:

Note: $\displaystyle\lim_{k\to\infty} \left(\frac{2^k+k^6}{\cancel{3^k}}\right)\Big/\left(\frac{2^k}{\cancel{3^k}}\right) = \lim_{k\to\infty} \frac{2^k+k^6}{2^k}$

$\displaystyle = \lim_{k\to\infty}\left(1+\frac{k^6}{2^k}\right)$ growth rates

$= 1$

is nonzero and finite so by LCT, $\sum \frac{2^k+k^6}{3^k}$ and $\sum \frac{2^k}{3^k} = \sum \left(\frac{2}{3}\right)^k$ either both converge or both diverge. Since $\sum\left(\frac{2}{3}\right)^k$ is a geometric series with $r<1$, it converges. Hence, $\boxed{\displaystyle\sum_{k=1}^{\infty}\frac{2^k+k^6}{3^k} \text{ converges}}$ as well.

Way 2: Root Test.

Since $\frac{2^k+k^6}{3^k} > 0$, use root test:

$\displaystyle L = \lim_{k\to\infty} \sqrt[k]{\frac{2^k+k^6}{3^k}}$

$\displaystyle = \lim_{k\to\infty} \sqrt[k]{\frac{2^k(1+k^6/2^k)}{3^k}}$

$\displaystyle = \lim_{k\to\infty} \sqrt[k]{\left(\frac{2}{3}\right)^k}\sqrt[k]{1+\frac{k^6}{2^k}}$

$\displaystyle = \lim_{k\to\infty} \frac{2}{3}\sqrt[k]{1+\underbrace{\frac{k^6}{2^k}}_{\to 0}}$

$= \frac{2}{3}$

0 by growth rates so $\sqrt[k]{1+\frac{k^6}{2^k}} \to 1$

$\boxed{\text{Since } L<1, \text{ the series converges by root test}}$

II. $\sum_{k=1}^{\infty}(-1)^k \frac{(k!)^2}{(2k+2)!}$. ← We want to use ratio test to deal with the factorials, but we need the summand to be positive!

Note: If you want to use the Alternating Series Test here, you must clearly explain why the assumptions hold, not just state them!

Check for absolute convergence:

$$\sum \left| (-1)^k \frac{(k!)^2}{(2k+2)!} \right| = \sum \frac{(k!)^2}{(2k+2)!}$$

We can now apply ratio test since the summand is positive:

$$L = \lim_{k\to\infty} \frac{[(k+1)!]^2}{[2(k+1)+2]!} \cdot \frac{(2k+2)!}{(k!)^2}$$

$$= \lim_{k\to\infty} \left[\frac{(k+1)!}{(k!)^2}\right]^2 \cdot \frac{(2k+2)!}{(2k+4)!}$$

$$= \lim_{k\to\infty} \left[\frac{(k+1)\cancel{k!}}{\cancel{k!}}\right]^2 \frac{\cancel{(2k+2)!}}{(2k+4)(2k+3)\cancel{(2k+2)!}}$$

$$= \lim_{k\to\infty} \frac{(k+1)^2}{(2k+4)(2k+3)}$$

$$= \frac{1}{4}.$$

Since $L<1$, the series $\sum \frac{(k!)^2}{(2k+2)!}$ converges by the ratio test. Hence, the original series converges absolutely and thus converges.

Bonus: [2 pts] Evaluate $\lim_{n\to\infty} \frac{(n!)^2}{(2n+2)!}$. To receive credit, you must justify your response!

A corollary to the divergence test ensures that if $\sum a_k$ converges then $\lim_{n\to\infty} a_n = 0$.

Since $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k+2)!}$ converges by our above work,

$$\lim_{n\to\infty} \frac{(n!)^2}{(2n+2)!} = 0.$$