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General Information:

This is a midterm from a previous semester. This means:

- This midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to this midterm.
- This midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take this exam:

You should treat this midterm should be as the actual exam. This means:

- "Practice like you play." Schedule 55 uninterrupted minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and projects with this intent, and there will be a problem on your midterm that will require you to apply the material in an unfamiliar setting.

How to use the solutions:

DO NOT JUST READ THE SOLUTIONS!!!

The least important aspect of the solutions is learning the steps necessary to solve a specific problem. You should be looking for the concepts required to provide solutions. Content may not be recycled, but concepts will be!

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

- -

Math 1152 Name: Solutions

Midterm 3 OSU Username (name.nn): __________

Autumn 2016 Lecturer: __________

Recitation Instructor: __________

Form A Recitation Time: __________

Instructions

- You have **55 minutes** to complete this exam. It consists of 6 problems on 12 pages including this cover sheet. Page 11 has possibly helpful formulas and may also be used for extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	16	
2	20	
3	15	
4	16	
5	15	
6	18	
Total	100	

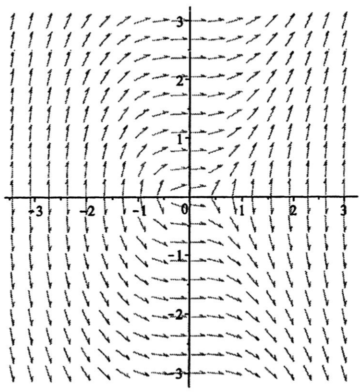
1. **Multiple Choice** [16 pts]

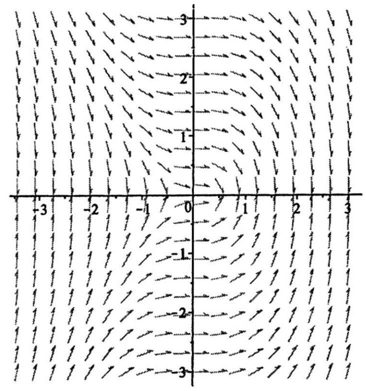
   Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

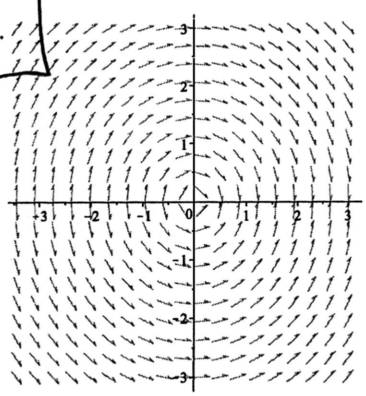
I. Which of the following is the direction field for the differential equation $\frac{dy}{dx} = \frac{x^2}{y}$? ← • In QI, $(x>0, y>0)$ we have $\frac{dy}{dx} > 0$ so the slopes are positive, which eliminates B, C.

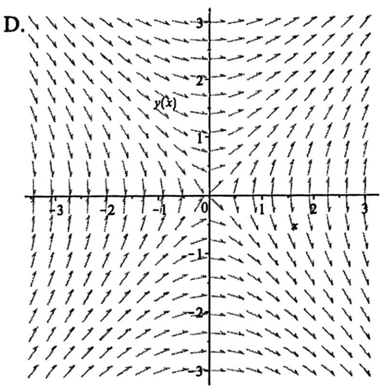
• In QIII $(x<0, y<0)$, $\frac{x^2}{y} < 0$ so the slopes of the tangent lines are negative, eliminating A.

→ The answer is C.

A. 

B. 

$\boxed{\text{C.}}$ 

D. 

II. Given that $\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$, find the fourth degree Taylor polynomial centered at $x = 0$ for $\ln(1+2x)$.

A. $p_4(x) = -2x - 2x^2 - \frac{8}{3}x^3 - 4x^4$

$\boxed{\text{B. } p_4(x) = 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4}$

C. $p_4(x) = -2x - x^2 - \frac{2}{3}x^3 - \frac{1}{2}x^4$

D. $p_4(x) = 2x - x^2 + \frac{2}{3} + x^3 - \frac{1}{2}x^4$

E. None of the above

Write out the first 4 terms in $\ln(1-x)$:

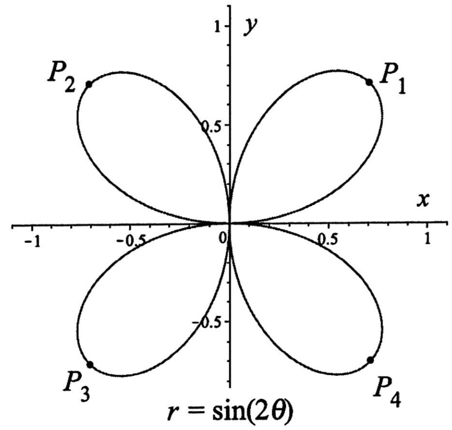
$$\ln(1-x) = -\frac{x^1}{1} - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$$

$$\ln(1+2x) = \ln(1-(-2x)) = -\frac{(-2x)}{1} - \frac{(-2x)^2}{2} - \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4}$$

$$= 2x - \frac{4x^2}{2} - \frac{-8x^3}{3} - \frac{16x^4}{4}$$

$$= 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \ldots$$

III. The curve C described by the polar equation $r = \sin(2\theta)$ is shown below:

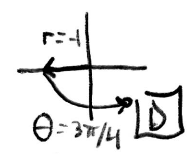


Which of the following represents the point on the curve when $\theta = \dfrac{3\pi}{4}$?

A. P_1 B. P_2 C. P_3 D. P_4

E. More than one of these F. None of these

Way1: When $\theta = \frac{3\pi}{4}$, $r = \sin(\frac{3\pi}{2}) = -1$. Draw →



Way2: Since $r = -1$, $\theta = \frac{3\pi}{4}$, we have

$$x = r\cos\theta = -1\left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} \rightarrow \boxed{D}$$
$$y = r\sin\theta = -1\left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2}$$

IV. Suppose $\sum_{k=0}^{\infty} a_k(x-1)^k$ diverges when $x = -2$. Then:

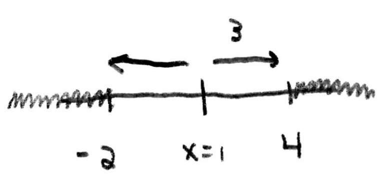
A. $\sum_{k=0}^{\infty} a_k$ must converge.

B. $\sum_{k=0}^{\infty} a_k$ must diverge.

C. $\sum_{k=0}^{\infty} a_k$ could converge or diverge.

D. None of the above.

The series diverges when $x=-2$, which is 3 units away from the center $x=1$: We thus only know that the series diverges if $x \le -2$ or $x \ge 4$. We know nothing about what the series does otherwise!



ex: • If $a_k = k!$, $\sum k!(x-1)^k$ diverges for all $x \neq 1$ by Ratio Test.

• If $a_k = \frac{1}{3^k}$, the ROC is 3.

2. **Short Answer** [20 pts]

   Answer each of the following questions and justify your responses unless otherwise requested.

   I. [6 pts] Find all values of $a > 0$ and a value for C such that $y(t) = C\cos(at)$ is a solution to the initial value problem:

$$\begin{cases} y''(t) + 4y(t) = 0 \\ y(0) = 3, \quad y'(0) = 0 \end{cases}$$

• Substitute into the ODE:

If $y = C\cos at$

$y' = -aC\sin at$

$y'' = -a^2C\cos at$

So $0 = y'' + 4y = -a^2C\cos at + 4C\cos at$

$0 = C\cos at\,[-a^2+4]$.

A solution to the IVP must be a function $y(t)$ for which the ODE holds for all $t \Rightarrow$ $C\cos at\,[-a^2+4] = 0$ for all t.

$\Rightarrow -a^2+4=0$

$a = \pm 2$.

Since $a>0$, we have $\boxed{a=2}$.

Check the IC: $y(0)=3 \rightarrow 3 = C\cos(2(0)) \rightarrow \boxed{C=3}$

$y'(t) = -2(3)\sin 2t$

$y'(0) = 0$ ✓.

   II. [4 pts] CIRCLE the correct response to each question. NO justification is necessary!

   Differentiating a power series __________ change the radius of convergence.

   MUST COULD $\boxed{\text{DOES NOT}}$

   Differentiating a power series __________ change the interval of convergence.

   MUST $\boxed{\text{COULD}}$ DOES NOT

III. [4 pts] Find all values of a such that the curve C described by the polar equation $r(\theta) = a - 3\cos\theta$ passes through the origin in the xy-plane.

The curve passes through the origin if there is a value of θ for which $r(\theta) = 0$ so set:

$$r(\theta) = a - 3\cos\theta = 0$$

$$\cos\theta = \frac{a}{3}.$$

The range of $\cos\theta$ is $[-1, 1]$, so the curve passes through the origin if:

$$-1 \le \frac{a}{3} \le 1$$

$$\boxed{-3 \le a \le 3}$$

IV. [6 pts] The curve C is described parametrically by:

$$\begin{cases} x(t) &= 1 - t^2 \\ y(t) &= t^4 \end{cases}, \quad t \ge 0$$

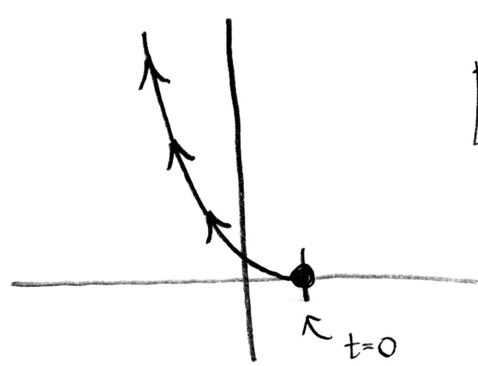
Eliminate the parameter to find a description in terms of x and y only. Sketch the curve and indicate the positive orientation on your sketch.

Note: $x = 1 - t^2 \rightarrow t^2 = 1 - x.$

$y = t^4 = (t^2)^2 = (1-x)^2$

The curve is part of the parabola $y = (1-x)^2$.

When $t = 0$, $x = 1$, $y = 0$. As t increases, y increases but x decreases, so we only obtain the left half of the parabola.



$$\boxed{C \text{ is } y = (1-x)^2, \ x \le 1}$$

3. [14 pts] A curve is described parametrically by:

$$\begin{cases} x(t) &= t^2 + 2t + 2 \\ y(t) &= 4t + 1 \end{cases}$$

for all t where $x(t)$ and $y(t)$ are well-defined.

I. Find $\frac{dy}{dx}$ in terms of t. Simplify your final answer!

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4}{2t+2} = \boxed{\frac{2}{t+1}}$$

II. Give the Cartesian equation(s) for all vertical tangent lines to the curve or state that there are none.

There is a vertical tangent line when $\frac{dy}{dx}$ is unbounded

$\rightarrow$ $t+1=0$ or $\boxed{t=-1}$

The vertical tangent line is $x = x(-1) = (-1)^2+2(-1)+2$

$\boxed{x=1}$

III. Find the Cartesian equation(s) for all tangent lines to the curve with slope 2.

- If the slope is 2; then $\frac{dy}{dx} = 2$

$\frac{2}{t+1} = 2 \rightarrow$ $2 = 2(t+1)$

$\cancel{2} = 2t + \cancel{2}$

$t = 0$

- $x(0) = (0)^2 + 2(0) + 2 \Rightarrow x = 2$

$y(0) = 4(0) + 1 \rightarrow y = 1$

So $y - y(0) = m[x - x(0)]$

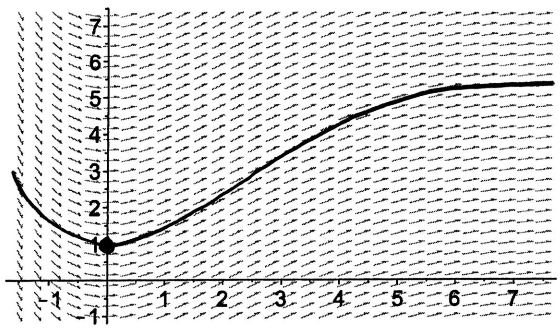
$y - 1 = 2[x-2]$.

$\boxed{y = 2x-3}$

4. [16 pts] Consider the initial value problem:

$$\frac{dy}{dx} = xe^{-x/2}, \qquad y(0) = 1.$$

I. The direction field for this differential equation is given below. Sketch the solution to the initial value problem. Make a conjecture about $\lim_{x\to\infty} y(x)$.



From the sketch, it looks like $\lim_{x\to\infty} y(x) =$ 5.

II. Calculate the specific solution to the initial value problem.

$$y = \int xe^{-x/2}\,dx$$

← DO NOT try anything other than integration by parts! If you did, what happens when you differentiate your answer?

Int. by parts: $u = x$ $\quad dv = e^{-x/2}dx$

$du = dx$ $\quad v = -2e^{-x/2}$

$$y = -2xe^{-x/2} - \int -2e^{-x/2}\,dx$$

$$y = -2xe^{-x/2} - 4e^{-x/2} + C.$$

$y(0)=1$: $\quad 1 = -2\overset{0}{\cancel{(0)}}e^0 - 4\overset{1}{\cancel{e^0}} + C \Rightarrow C = 5$.

So $\boxed{y(x) = -2xe^{-x/2} - 4e^{-x/2} + 5}$

III. From your solution, compute $\lim_{x\to\infty} y(x)$.

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty}\left[-2xe^{-x/2} - 4e^{-x/2} + 5\right]$$

$$= \lim_{x\to\infty}\left[-\frac{2x}{e^{x/2}} - \frac{4}{e^{x/2}} + 5\right]$$

↓ 0 by growth rates (or L'Hopital).

$$= 0 + 0 + 5$$

$\boxed{\lim_{x\to\infty} y(x) = 5}$, consistent with the picture!

5. [15 pts] Consider the function $y = \dfrac{x^2}{1+2x}$.

I. Give the Taylor series centered at $x = 0$ in summation notation for this function. What is the radius of convergence of this series?

$$y = \frac{x^2}{1+2x} = x^2\left[\frac{1}{1+2x}\right].$$

Note: $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad |x|<1$

$$x^2\left[\frac{1}{1+2x}\right] = x^2\left[\frac{1}{1-(-2x)}\right] = x^2\sum_{k=0}^{\infty}(-2x)^k, \quad |-2x|<1$$

$$= \boxed{\sum_{k=0}^{\infty}(-1)^k 2^k x^{k+2}, \quad |x|<\tfrac{1}{2}}$$

↑(ROC is $\frac{1}{2}$).

II. Use the Taylor series you found in Part I to calculate $y'''(0)$.

Write out the series:

$$y = x^2 - 2x^3 + 4x^4 - \ldots$$
$$y' = 2x - 6x^2 + 16x^3 - \ldots$$
$$y'' = 2 - 12x + 48x^2 - \ldots$$
$$y''' = -12 + 48x - \ldots \xrightarrow[\text{plug in } x=0]{} \boxed{y'''(0) = -12}$$

III. Give the sum of the first 3 nonzero terms[1] in the Taylor series centered at $x = 0$ for

$$f(x) = \int_0^x \frac{t^2}{1+2t}\,dt.$$

We found above $\frac{t^2}{1+2t} = t^2 - 2t^3 + 4t^4 - \ldots$

Integrate: $\int_0^x \frac{t^2}{1+2t}\,dt = \int_0^x (t^2 - 2t^3 + 4t^4 - \ldots)\,dt$

$$= \left[\tfrac{1}{3}t^3 - \tfrac{1}{2}t^4 + \tfrac{4}{5}t^5 - \right|_0^x$$

$$= \boxed{\tfrac{1}{3}x^3 - \tfrac{1}{2}x^4 + \tfrac{4}{5}x^5 - \ldots}$$

[1] *For a Taylor series $h(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$, the phrase "the sum of the first n nonzero terms" means the sum of the first n powers (in ascending order) of $(x-c)$ whose coefficients are nonzero.*

6. [18 pts] The power series for a certain function $f(x)$ is given by:

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{(x+2)^k}{3^k}.$$

I. Find the interval of convergence of the power series.

1. Use Ratio test to find ROC:

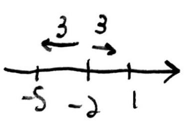
$$L(x) = \lim_{k\to\infty} \left| \frac{\cancel{(-1)^{k+1}}}{k+1} \cdot \frac{(x+2)^{k+1}}{3^{k+1}} \cdot \frac{k}{\cancel{(-1)^k}} \frac{3^k}{(x+2)^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{k}{k+1} \cdot \frac{(x+2)^{k+1}}{(x+2)^k} \cdot \frac{3^k}{3^{k+1}} \right|$$

$$= \frac{|x+2|}{3} \lim_{k\to\infty} \frac{k}{k+1}$$

$$L(x) = \frac{|x+2|}{3}$$

DO **NOT** say $|x+2|<3 \to |x|<1$! The absolute value is **NOT** linear!



- The series converges when $L(x)<1 \to |x+2|<3. \to$ $\boxed{\text{The ROC is } 3}$
- The series is centered at $x=-2$, so it will converges when $-5<x<1$.
- $f(1) = \sum \frac{(-1)^k}{k} \frac{\cancel{3^k}}{\cancel{3^k}}$ converges by the alt. series test
- $f(-5) = \sum \frac{(-1)^k}{k} \frac{(-3)^k}{3^k} = \sum \frac{(-1)^k}{k} \frac{(-1)^k \cancel{3^k}}{\cancel{3^k}} = \sum \frac{1}{k}$ which diverges (it is the harmonic series).

$\to$ $\boxed{\text{IOC is } -5<x\leq 1 \text{ or } [-5,1]}$

II. Let $g(x) = f(x-2)$. Calculate $\lim_{x\to 0} \frac{g(x)}{\sin x}$.

Hint: The power series for $g(x)$ is centered at $x=0$.

$$g(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{[(x\cancel{-2})\cancel{+2}]^k}{3^k}$$

$$= -\frac{1}{3}x + \frac{1}{18}x^2 - \frac{1}{81}x^3 + \ldots$$

So, $\lim_{x\to 0} \frac{g(x)}{\sin x} = \lim_{x\to 0} \frac{-\frac{1}{3}x + \frac{1}{18}x^2 - \frac{1}{81}x^3 + \ldots}{x - \frac{1}{6}x^3 + \ldots}$

$$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}} \left[\frac{-1/3 + 1/18\, x - 1/81\, x^2 + \ldots}{1 - 1/6\, x^2 + \ldots} \right]$$

$$= \boxed{-1/3}$$