DISCLAIMER

General Information:

This is a midterm from a previous semester. This means:

- This midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to this midterm.
- This midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take this exam:

You should treat this midterm should be as the actual exam. This means:

- "Practice like you play." Schedule 55 uninterrupted minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and projects with this intent, and there will be a problem on your midterm that will require you to apply the material in an unfamiliar setting.

How to use the solutions:

DO NOT JUST READ THE SOLUTIONS!!!

The least important aspect of the solutions is learning the steps necessary to solve a specific problem. You should be looking for the concepts required to provide solutions. Content may not be recycled, but concepts will be!

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

Math 1152 | Name: Solutions

Midterm 2 | OSU Username (name.nn): ____________

Spring 2016 | Lecturer: ____________

Recitation Instructor: ____________

Form A | Recitation Time: ____________

Instructions

- You have **55 minutes** to complete this exam. It consists of 6 problems on 10 pages including this cover sheet. Page 10 has possibly helpful formulas and may also be used for extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	12	
2	18	
3	15	
4	24	
5	14	
6	17	
Total	100	

1. **Multiple Choice** [12 pts]

   Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

I. Given that $\displaystyle\sum_{k=1}^{\infty} \frac{12}{k^2+2k} = 9$, find $\displaystyle\sum_{k=2}^{\infty} \frac{12}{k^2+2k}$.

A. 4 $\boxed{\text{B. } 5}$ C. 9 D. 13 E. None of the above

$$\sum_{k=1}^{\infty} \frac{12}{k^2+2k} = \frac{12}{(1)^2+2(1)} + \sum_{k=2}^{\infty} \frac{12}{k^2+2k}$$

$$9 = 4 + \sum_{k=2}^{\infty} \frac{12}{k^2+2k} \rightarrow \sum_{k=2}^{\infty} \frac{12}{k^2+2k} = 5.$$

II. Which of the following functions is a general solution to the differential equation:

$$\frac{dy}{dt} + 2y = 2?$$

A. $y(t) = e^{-2t} + C$ $\boxed{\text{B. } y(t) = Ce^{-2t} + 1}$ C. $y(t) = t^2 + C$

D. $y(t) = C\cos(2t) + 2$ E. $y(t) = C\sin(2t) + 2$ F. None of the above

For each choice, calculate $\frac{dy}{dt}$ and substitute:

A. $\frac{dy}{dt} = -2e^{2t}$

$\frac{dy}{dt} + 2y = -2e^{2t} + 2(e^{-2t}+C) = 2C \neq 2.$

$\rightarrow$ Not a soln.

B: $y'(t) = -2Ce^{-2t}$

$\frac{dy}{dt} + 2y = -2Ce^{-2t} + 2(Ce^{-2t}+1)$
$= 2$ ✓.

III. To which of the following series could the integral test be applied?

A. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ B. $\displaystyle\sum_{k=1}^{\infty} \sec^2 k$ C. $\displaystyle\sum_{k=1}^{\infty} k\sin(k^2)$

$\boxed{\text{D. } \displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}}$ E. More than one of these F. None of these

to Σa_k

To apply integral test, you need to find $f(x)$ so $f(k) = a_k$ and have $f(x)$ be continuous, decreasing, and positive.

- $f(x) = \frac{(-1)^x}{x}$ is not continuous nor decreasing
- $f(x) = \sec^2 x$ is not continuous, decreasing, or strictly positive

$x \sin x^2$

2. **Short Answer** [18 pts]

Answer the following questions. Each question is worth 3 points. You do not need to justify your answer. There is no partial credit and no penalty for guessing.

I. Suppose that $\{a_n\}_{n\geq 1}$ is a sequence such that $s_n = \dfrac{n^3}{6-n^2}$, where $s_n = \displaystyle\sum_{k=1}^{n} a_k$ for $n \geq 1$.

A. $a_3 =$ $\underline{-13}$.

$$a_3 = s_3 - s_2 = \frac{(3)^3}{6-(3)^2} - \frac{(2)^3}{6-(2)^2} = \frac{27}{-3} - \frac{8}{2} = -13$$

B. $a_1 + a_2 + a_3 =$ $\underline{-9}$.

$a_1 + a_2 + a_3 = s_3$ by definition

$= -9$ by above calculations.

C. Determine whether $\displaystyle\lim_{n\to\infty} s_n$ exists. If it does, give its value.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n^3}{6-n^2} = \infty \text{ , so } \boxed{\lim_{n\to\infty} s_n \text{ DNE}}$$

D. Determine whether $\displaystyle\sum_{k=1}^{\infty} a_k$ converges or diverges. If it converges, give the value to which it converges.

Since $\displaystyle\lim_{n\to\infty} s_n$ DNE, $\boxed{\displaystyle\sum_{k=1}^{\infty} a_k \text{ diverges}}$

II. Give the general partial fraction decomposition for $\dfrac{2x+1}{x^3+x^2}$. DO NOT SOLVE FOR THE CONSTANTS!

$$\frac{2x+1}{x^3+x^2} = \frac{2x+1}{x^2(x+1)} = \boxed{\frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+1}}$$

III. For which values of b does the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{2b}}$ converge?

$\sum \frac{1}{k^p}$ converges if $p > 1$ so we need $2b > 1$

$$\boxed{b > \tfrac{1}{2}}$$

3. [15 pts] Given that $\dfrac{5}{(2x-1)(x^2+1)} = \dfrac{4}{2x-1} - \dfrac{2x+1}{x^2+1}$, determine if the improper integral:

$$\int_1^\infty \frac{5}{(2x-1)(x^2+1)}\,dx$$

converges or diverges. If it converges, give the value to which it converges and **simplify** your final answer. Make sure you use proper notation in your solution!

Note: $\displaystyle\int \frac{5}{(2x-1)(x^2+1)}\,dx = \int\left[\frac{4}{2x-1} - \frac{2x+1}{x^2+1}\right]dx$

$$= \int\left[\frac{4}{2x-1} - \frac{2x}{x^2+1} - \frac{1}{x^2+1}\right]dx$$

$$= 2\ln|2x-1| - \ln|x^2+1| - \arctan x + C$$

So, $\displaystyle\int_1^\infty \frac{5}{(2x-1)(x^2+1)}\,dx = \lim_{b\to\infty}\left[2\ln|2x-1| - \ln|x^2+1| - \arctan x\right|_1^b$

↑ $\infty - \infty$ is NOT defined! Combine $\ln(\ )$ terms!

$$= \lim_{b\to\infty}\left[\ln(2x-1)^2 - \ln(x^2+1) - \arctan x\right|_1^b$$

$$= \lim_{b\to\infty}\left[\ln\frac{(2x-1)^2}{x^2+1} - \arctan x\right|_1^b$$

$$= \lim_{b\to\infty}\left\{\left[\ln\frac{(2b-1)^2}{b^2+1} - \arctan b\right] - \left[\ln\frac{1}{2} - \arctan 1\right]\right\}$$

$$= \ln 4 - \frac{\pi}{2} - \ln\frac{1}{2} + \frac{\pi}{4}$$

$$= \boxed{\ln 8 - \frac{\pi}{4}}$$

4. [24 pts] Calculate the following antiderivatives and **simplify** your final answers as completely as possible.

I. [12 pts] $\int \sin^3(2\theta)\cos^2(2\theta)\,d\theta$.

- If $u = \sin 2\theta$, save a copy of $u' = 2\cos 2\theta$ for du. But, we'd only have 1 copy of $\cos 2\theta$ left! This is a bad choice!
- If $u = \cos 2\theta$, save a copy of $u' = -2\sin 2\theta$ for du. We'd have 2 copies of $\sin 2\theta$ left, which could easily be converted!

$u = \cos 2\theta$

$du = -2\sin 2\theta\, d\theta$

$\dfrac{du}{-2\sin 2\theta} = d\theta$

So $\displaystyle\int \sin^3 2\theta \cos^2 2\theta\, d\theta = \int \sin^3 2\theta \cdot u^2 \frac{du}{-2\sin 2\theta}$

$= -\frac{1}{2}\int \sin^2 2\theta\, u^2\, du$ ← convert $\sin 2\theta$ to $\cos 2\theta$

$= -\frac{1}{2}\int (1-\cos^2 2\theta)\, u^2\, du$

$= -\frac{1}{2}\int (1-u^2)\, u^2\, du$

$= -\frac{1}{2}\int (u^2 - u^4)\, du$

$= -\frac{1}{2}\cdot\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right] + C$

$= \boxed{-\frac{1}{6}\cos^3 2\theta + \frac{1}{10}\cos^5 2\theta + C}$

II. [12 pts] $\displaystyle\int \frac{4}{(x^2+36)^{3/2}}\,dx.$

$\uparrow$ $u^2 + a^2 \to \quad u = a\tan\theta$. Here $u^2 = x^2 \to u = x$

$a^2 = 36 \to a = 6$

$$x = 6\tan\theta$$

$$dx = 6\sec^2\theta\, d\theta.$$

So
$$\int \frac{4}{(x^2+36)^{3/2}}\,dx = \int \frac{4}{(36\tan^2\theta+36)^{3/2}} \cdot 6\sec^2\theta\, d\theta$$

$$= \int \frac{4}{(36\sec^2\theta)^{3/2}} \cdot 6\sec^2\theta\, d\theta$$

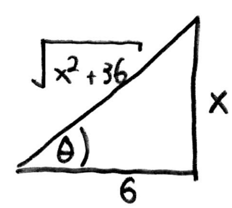
$$= \int \frac{4}{216\sec^3\theta} \cdot 6\sec^2\theta\, d\theta$$

$$= \frac{1}{9}\int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{9}\int \cos\theta\, d\theta.$$

$$= \frac{1}{9}\sin\theta + C$$

$$= \boxed{\frac{1}{9}\,\frac{x}{\sqrt{x^2+36}} + C}$$



$x = 6\tan\theta$

$\tan\theta = \frac{x}{6}$

5. [14 pts] Determine if the following series converge or diverge and **justify** your response!

I. $\sum_{k=1}^{\infty} \cos\left(\frac{k+1}{k}\right)$.

Try divergence test: $\lim_{k\to\infty} \frac{k+1}{k} = 1$

so $\lim_{k\to\infty} \cos\left(\frac{k+1}{k}\right) = \cos 1$

Since $\cos 1 \neq 0$, the series diverges by divergence test!

II. $\sum_{k=2}^{\infty} 2^{3-2k}$.

Note: $2^{3-2k} = 2^3 \cdot 2^{-2k} = \frac{8}{2^{2k}} = \frac{8}{(2^2)^k} = \frac{8}{4^k} = 8\left(\frac{1}{4}\right)^k$.

So, $\sum_{k=2}^{\infty} 2^{3-2k} = \sum_{k=2}^{\infty} 8\left(\frac{1}{4}\right)^k$.

This is a geometric series with $r = \frac{1}{4} < 1$, so it converges!

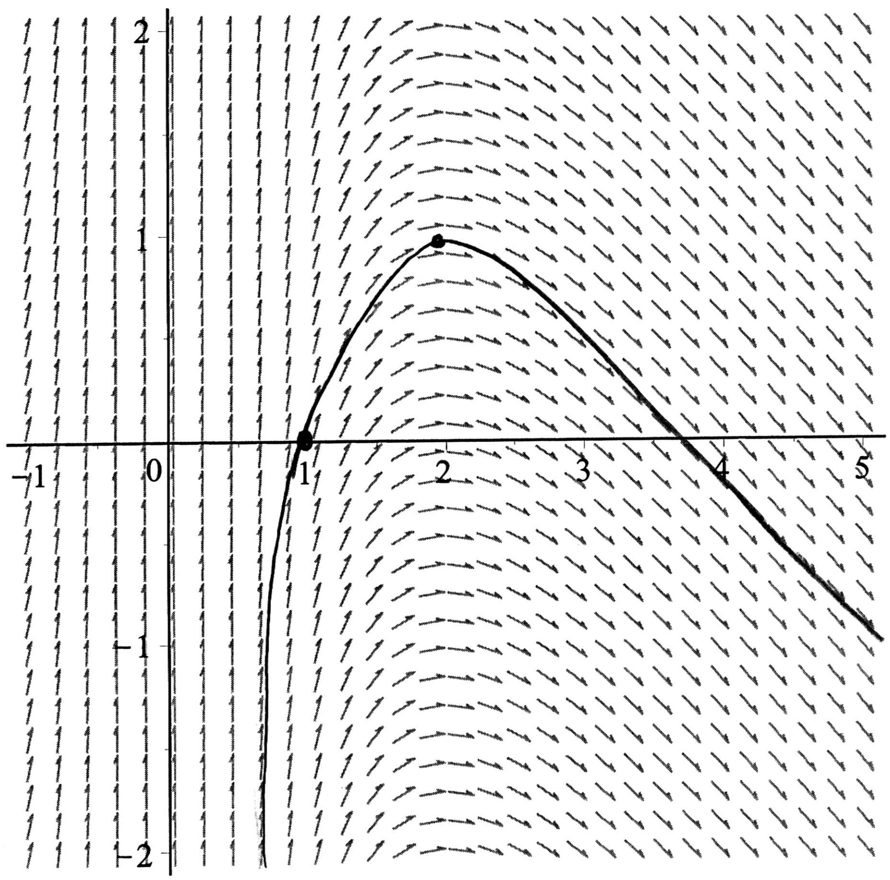
6. [16 pts] Consider the initial value problem:

$$\frac{dy}{dt} = \frac{4 - t^2}{t^2}, \qquad y(1) = 0.$$

I. [1 pt] Is the differential equation linear or nonlinear?

$\boxed{\text{Linear}}$ Note, linear vs nonlinear refers to the dependant variable y!

II. The direction field for this differential equation is shown below:



A. [2 pts] Sketch the solution to the initial value problem on the direction field for $t > 0$.

B. [1 pts] Fill in the blanks below:

From your sketch, it appears that y has a maximum value of $y =$ __1__ when $t =$ __2__.

(Give the approximate values from your sketch!)

III. [8 pts] Find the specific solution to the initial value problem

$$\frac{dy}{dt} = \frac{4 - t^2}{t^2}, \qquad y(1) = 0.$$

$$\frac{dy}{dt} = \frac{4}{t^2} - \frac{t^2}{t^2} = 4t^{-2} - 1$$

$$y = \int (4t^{-2} - 1)\, dt$$

$$y = -4t^{-1} - t + C \quad \leftarrow \text{General solution}$$

Using $y(1) = 0$: $\quad 0 = -4(1)^{-1} - 1 + C$

$$C = 5$$

So: $\quad y = -4t^{-1} - t + 5$

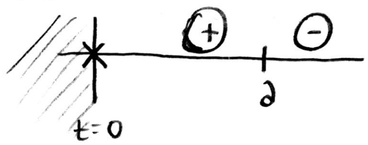
$$\boxed{y = -\frac{4}{t} - t + 5}$$

IV. [2 pts] Find the time t, with $t > 0$, when y attains its maximum. Justify your answer!

Hint: What is true about $y'(t)$ when $y(t)$ has a maximum?

Note $\frac{dy}{dt} = \frac{4-t^2}{t^2} = 0$ when $4 - t^2 = 0 \rightarrow t = \pm 2$.

Sign chart for y':



We see $t = 2$ corresponds to a relative maximum. This is an absolute maximum as well!

V. [2 pts] Find the maximum value for $y(t)$.

$$y(2) = -\frac{4}{2} - 2 + 5$$

$$\boxed{y(2) = 1.}$$

The maximum value of $y(t)$ for $t > 0$ is 1.

* Compare to our sketch!