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General Information:

This is a midterm from a previous semester. This means:

- This midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to this midterm.
- This midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take this exam:

You should treat this midterm should be as the actual exam. This means:

- "Practice like you play." Schedule 55 uninterrupted minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and projects with this intent, and there will be a problem on your midterm that will require you to apply the material in an unfamiliar setting.

How to use the solutions:

DO NOT JUST READ THE SOLUTIONS!!!

The least important aspect of the solutions is learning the steps necessary to solve a specific problem. You should be looking for the concepts required to provide solutions. Content may not be recycled, but concepts will be!

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

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Math 1152 Name: Solutions

Midterm 3 OSU Username (name.nn): ______________

Spring 2016 Lecturer: ______________

Recitation Instructor: ______________

Form A Recitation Time: ______________

Instructions

- You have **55 minutes** to complete this exam. It consists of 5 problems on 12 pages including this cover sheet. Page 11 has possibly helpful formulas and may also be used for extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	12	
2	18	
3	20	
4	20	
5	30	
Total	100	

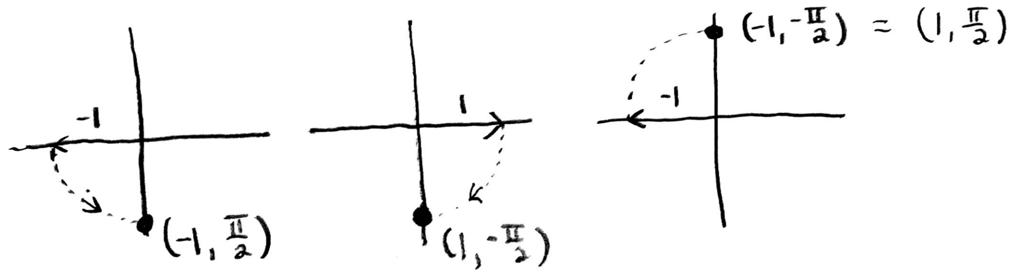
1. **Multiple Choice** [12 pts]

Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

I. A point in the xy-plane is described by the *polar coordinates* $(r, \theta) = \left(1, \frac{\pi}{2}\right)$.

Which of the following gives an alternate description of the point in polar coordinates?

A. $\left(-1, \frac{\pi}{2}\right)$ B. $\left(1, -\frac{\pi}{2}\right)$ C. $\left(-1, -\frac{\pi}{2}\right)$ D. None of the above



II. Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{2k^2}$. This series:

A. Converges absolutely

B. Converges conditionally

C. Diverges

D. None of the above

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{2k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{2k^2}$$ ← This is a p-series w/ $p = 2 > 1$ so it converges!

Since $\sum \left| \frac{(-1)^k}{2k^2} \right|$ converges, $\sum \frac{(-1)^k}{2k^2}$ converges absolutely.

III. Which of the following is <u>**necessary**</u> for the Ratio Test to be applied to $\sum_{k=0}^{\infty} a_k$?

A. $a_k \geq 0$ for all k eventually.

B. a_k is decreasing for all k eventually.

C. $\lim_{k \to \infty} a_k = 0$.

D. $\sum_{k=0}^{\infty} a_k$ is an alternating series.

E. More than one of these

F. None of these

2. **Multiselect** [18 pts]

Directions: Each problem below is worth 9 points. Circle *all* of the responses that MUST be true for each problem below. Note that there may be more than one correct response or even no correct responses!

A perfect answer for each part is worth 9 points. If you circle an incorrect choice, you will be penalized 3 points. If you do not circle a correct choice, you will be penalized 3 points. However, you cannot score below a 0 for either part of this problem. Thus, the possible scores for each part are 0, 3, 6, or 9 points.

I. [9 pts] Suppose the curve C is defined parametrically by:

$$\begin{cases} x(t) &= 2t \\ y(t) &= t^3 - 3t \end{cases}, \quad t \geq 0$$

CIRCLE *all* of the following statements that MUST be true.

A. $\dfrac{dy}{dx}$ increases as t increases.

B. The curve has no vertical tangent lines.

C. The point $(2, -2)$ is on the curve.

D. The curve has no horizontal tangent lines.

E. $\dfrac{dy}{dx} = -\dfrac{3}{2}$ when $x = 0$.

F. There is a time $t \geq 0$ when $\dfrac{dy}{dx} = 1$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2}$$

A <u>TRUE</u>; note that this depends on the fact $t \geq 0$!

B. <u>TRUE</u>; $\frac{dy}{dx}$ is never unbounded in finite time

C. <u>TRUE</u>; When $x=2$, we find $x = 2t = 2 \to t=1$.
When $t=1$, $y(1) = (1)^3 - 3(1) = -2$.

D. <u>FALSE</u>; $dy/dx = 0$ when $3t^2-3=0 \to t=+1$.

E. <u>TRUE</u>; When $x=0$, $2t = 0 \to t=1$. $\left.\frac{dy}{dx}\right|_{t=0} = \frac{3(0)^2-3}{2} = -\frac{3}{2}$

F. <u>TRUE</u>; $\frac{dy}{dx} = \frac{3t^2-3}{2} = 1$

$3t^2-3=2$

$3t^2=5 \leftarrow$ This has a solution!

II. [9 pts] The first four nonzero terms in the Taylor series for a certain function $f(x)$ centered at $x = 0$ are:

$$f(x) = 3 + 2x - 5x^3 + x^5 + \dots$$

Suppose it is also known that the series for $f(1)$ converges.

CIRCLE *all* of the following statements that MUST be true.

A. $f(0) = 3$

B. $f''(0) = 0$

C. $f'''(0) = -5$

D. $f(2x) = 6 + 4x - 40x^3 + 32x^5 + \dots$

E. The series for $f(-1)$ must converge.

F. The series for $f'(1/2)$ must converge.

A. TRUE; $f(0) = 3 + 2(0) - \dots = 3.$

B. TRUE; $f'(x) = 2 - 15x^2 + 5x^4 + \dots$

$f''(x) = -30x + 20x^3 + \dots$

So $f''(0) = 0$

C. FALSE; $f'''(x) = -30 + 60x^2 + \dots$

$f'''(0) = -30.$

D. FALSE; $f(2x) = 3 + 2(2x) - 5(2x)^3 + (2x)^5$

$= \boxed{3} + 4x - 40x^3 + 32x^5 + \dots$

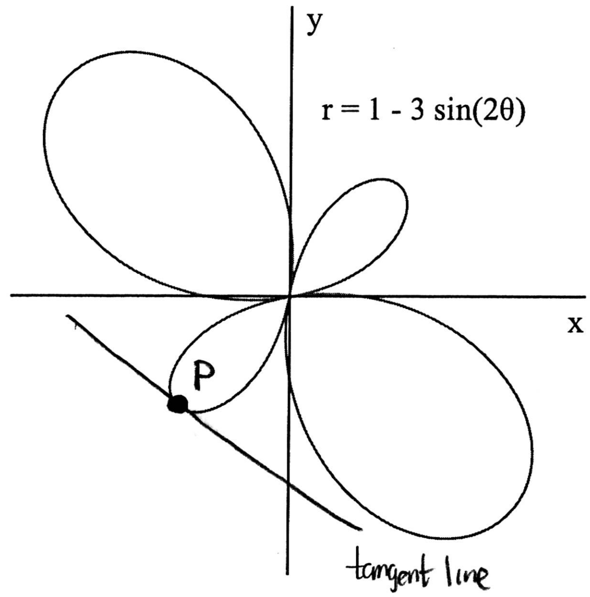
!

E. FALSE; $f(1)$ converges. Since $x=1$ is 1 unit away from the center $x=0$, the ROC is at least 1, but this does NOT mean $f(-1)$ converges!

For example, $f(x) = 3 + 2x - 5x^3 + x^5 + \sum_{k=6}^{\infty} \frac{(-1)^k x^k}{k}$

F. TRUE; the radius of convergence of $f(x)$ is at least 1. Since differentiating does NOT change the ROC, and $x = \frac{1}{2}$ is $\frac{1}{2}$ away from the center $x=0$, the series for $f'(\frac{1}{2})$ conv.

3. [20 pts] The curve described by the polar equation $r = 1 - 3\sin(2\theta)$ is shown below:



Note $r<0$ when $\theta = \frac{\pi}{4}$!

I. On the figure plot the point on the curve when $\theta = \dfrac{\pi}{4}$. Label it P.

II. On the figure, draw the tangent line to the curve at P.

III. Find the Cartesian coordinates (x, y) of the point on the curve when $\theta = \dfrac{\pi}{4}$.

When $\theta = \frac{\pi}{4}$:

- $r = 1 - 3\sin\left(\frac{\pi}{2}\right) = \underline{-2}$

- $x = r\cos\theta = -2\cos\frac{\pi}{4}$

  $\rightarrow \boxed{x\left(\frac{\pi}{4}\right) = -\sqrt{2}}$

- $y = r\sin\theta = -2\sin\frac{\pi}{4}$

  $\rightarrow \boxed{y\left(\frac{\pi}{4}\right) = -\sqrt{2}}$

* If you were confused by a negative r value in I, note that after doing III, you should be able to plot P correctly!
If you didn't compute r at all in I, BE CAREFUL! $(-r, \theta)$ is very different from $(+r, \theta)$!

IV. Find $\dfrac{dx}{d\theta}$ when $\theta = \dfrac{\pi}{4}$.

$$x = r\cos\theta = (1 - 3\sin 2\theta)\cos\theta.$$

$$\frac{dx}{d\theta} = -6\cos 2\theta\cos\theta - (1 - 3\sin 2\theta)\sin\theta$$

$$\left.\frac{dx}{d\theta}\right|_{\theta=\frac{\pi}{4}} = -6\overset{0}{\cancel{\cos}}\tfrac{\pi}{2}\cos\tfrac{\pi}{4} - \left(1 - 3\sin\tfrac{\pi}{2}\right)\sin\tfrac{\pi}{4}$$

$$= \boxed{\sqrt{2}}$$

V. Given that $\dfrac{dy}{d\theta} = -\sqrt{2}$, find the slope of the tangent line to the curve when $\theta = \dfrac{\pi}{4}$.

$$m_{tan} = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \quad \text{so} \quad \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \frac{-\sqrt{2}}{\sqrt{2}}$$

$$\boxed{m_{tan} = -1}$$

VI. Find the Cartesian description of the tangent line to the curve at $\theta = \dfrac{\pi}{4}$.

Express your final answer in the form $y = mx + b$.

$$y - y\left(\tfrac{\pi}{4}\right) = m_{tan}\left[x - x\left(\tfrac{\pi}{4}\right)\right]$$

$$y - (-\sqrt{2}) = -1\left[x - (-\sqrt{2})\right]$$

$$y + \sqrt{2} = -x - \sqrt{2}$$

$$\boxed{y = -x - 2\sqrt{2}}$$

4. [20 pts] Determine whether each series below converges or diverges. In order to get full credit, you must *fully* justify your work by:

- **Stating** any convergence test you use and why the test applies.
- **Explaining** the conclusions of the test!

  You may quote any results about p-series or geometric series, but you must *clearly* explain why the type of series you are considering converges or diverges!

I. $\sum_{k=0}^{\infty} \left[\frac{3k}{4k+1}\right]^k$

Since $\left(\frac{3k}{4k+1}\right)^k \geq 0$ for all k, we apply the Root test:

$$L = \lim_{k\to\infty} \sqrt[k]{\left(\frac{3k}{4k+1}\right)^k} = \lim_{k\to\infty} \frac{3k}{4k+1}$$

$$= \frac{3}{4}.$$

Since $L < 1$, the series converges by the Root Test.

II. $\sum_{k=1}^{\infty} \frac{\cos(2k)}{k^2}$ ← We cannot apply comparison tests to this since $\frac{\cos 2k}{k^2}$ is NOT strictly positive for all large k!

→ Check for absolute convergence:

For $\sum_{k=1}^{\infty} \left|\frac{\cos 2k}{k^2}\right|$, note $\left|\frac{\cos 2k}{k^2}\right| \geq 0$, so we apply the comparison test with $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

Note: 1. $\left|\frac{\cos 2k}{k^2}\right| \leq \frac{1}{k^2}$ for all k

2. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges since it is a p-series with $p>1$.

Hence $\sum \left|\frac{\cos 2k}{k^2}\right|$ converges by the comparison test.

and since $\sum \left|\frac{\cos 2k}{k^2}\right|$ converges, $\sum \frac{\cos 2k}{k^2}$ converges.

5. [25 pts] (Taylor Series)

I. A. Write out the first 3 nonzero terms in the Taylor series centered at $x = 0$ for:

$$y = 6\cos(x^3).$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots$$

$$\cos(x^3) = 1 - \frac{1}{2}(x^3)^2 + \frac{1}{24}(x^3)^4 - \dots$$

$$6\cos(x^3) = \boxed{6 - 3x^6 + \frac{1}{4}x^{12} - \dots}$$

B. Write out the first 3 nonzero terms in the Taylor series centered at $x = 0$ for

$$f(x) = \int_0^x \cos(t^3)\,dt.$$

$$f(x) \approx \int_0^x \left[6 - 3t^6 + \frac{1}{4}t^{12} - \dots\right] dt$$

$$= 6t - \frac{3}{7}t^7 + \frac{1}{52}t^{13} \Big|_0^x$$

$$= \boxed{6x - \frac{3}{7}x^7 + \frac{1}{52}x^{13}} - \dots$$

C. Evaluate the limit below using Taylor series:

$$\lim_{x\to 0} \frac{\cos(x^3) - 1}{x^6}$$

$$= \lim_{x\to 0} \frac{\left[\cancel{6} - 3x^6 + \frac{1}{4}x^{12} - \dots\right] - \cancel{6}}{x^6}$$

$$= \lim_{x\to 0} \frac{\cancel{x^6}\left(-3 + \frac{1}{4}x^6 - \dots\right)}{\cancel{x^6}}$$

$$= \boxed{-3}$$

II. The power series for a certain function $f(x)$ is given by $f(x) = \sum_{k=1}^{\infty} \frac{(x+3)^k}{k}$.

A. Find the radius of convergence of the power series.

$$L = \lim_{k\to\infty} \left| \frac{(x+3)^{k+1}}{k+1} \cdot \frac{k}{(x+3)^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{(x+3)^{k+1}}{(x+3)^k} \cdot \frac{k}{k+1} \right|$$

$$= |x+3| \lim_{k\to\infty} \frac{k}{k+1}$$

$$= |x+3|$$

The series converges when $|x+3| < 1$ $\rightarrow$ $\boxed{\text{ROC is } 1}$

B. Give the interval of convergence for the power series.

The series converges for $-4 < x < -2$.

At $x=-4$: $f(-4) = \sum_{k=1}^{\infty} \frac{(-4+3)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$

This is alternating! Also, $\frac{1}{k}$ decreases and $\lim_{k\to\infty} \frac{1}{k} = 0$

So the series converges by alt series test.

At $x=-2$: $f(-2) = \sum_{k=1}^{\infty} \frac{(-2+3)^k}{k} = \sum_{k=1}^{\infty} \frac{(1)^k}{k} = \sum_{k=1} \frac{1}{k}$.

This diverges as it is the harmonic series.

$\boxed{\text{The IOC is thus } -4 \le x < -2 \text{ or } [-4,-2)}$