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General Information:

This is a midterm from a previous semester. This means:

- This midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to this midterm.
- This midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take this exam:

You should treat this midterm should be as the actual exam. This means:

- "Practice like you play." Schedule 55 uninterrupted minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and projects with this intent, and there will be a problem on your midterm that will require you to apply the material in an unfamiliar setting.

How to use the solutions:

DO NOT JUST READ THE SOLUTIONS!!!

The least important aspect of the solutions is learning the steps necessary to solve a specific problem. You should be looking for the concepts required to provide solutions. Content may not be recycled, but concepts will be!

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

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Math 1172

Midterm 2

Autumn 2016

Form A

Name: Solutions

OSU Username (name.nn): __________

Lecturer: __________

Recitation Instructor: __________

Recitation Time: __________

Instructions

- You have **55 minutes** to complete this exam. It consists of 5 problems on 12 pages including this cover sheet. Page 11 has possibly helpful formulas and pages 11 and 12 may also be used for extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work!** Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	20	
2	16	
3	16	
4	18	
5	30	
Total	100	

1. **Multiple Choice** [20 pts]

Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

I. Suppose that $\{a_k\}$ is a sequence for which $a_k > 0$ for all $k \geq 1$ and

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{1}{2}.$$

Then, the Ratio Test guarantees:

Ratio Test ensures the series converges if $L = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$ but does NOT give info about the value to which series converges

A. $\sum_{k=1}^{\infty} a_k$ diverges.

B. $\sum_{k=1}^{\infty} a_k$ converges to $\frac{1}{2}$.

C. $\sum_{k=1}^{\infty} a_k$ converges but there is not enough information to determine its value.

D. Nothing; the Ratio Test may not apply since we do not know a formula for a_k.

II. Given that $f(x) = \sum_{k=0}^{\infty} \frac{2^k}{3k+1}(x-2)^k$, find $f(2)$.

A. 2 | B. 1 | C. $\frac{4}{7}$ | D. 0 | E. None of the above.

Write out a few terms:

$$f(x) = 1 + \frac{1}{2}(x-2) + \frac{4}{7}(x-2)^2 + \dots \rightarrow \boxed{f(2) = 1}$$

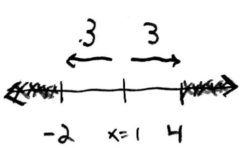
III. Suppose $\sum_{k=0}^{\infty} a_k(x-1)^k$ diverges when $x = -2$. Then:

A. $\sum_{k=0}^{\infty} a_k$ must converge.

B. $\sum_{k=0}^{\infty} a_k$ must diverge.

C. $\sum_{k=0}^{\infty} a_k$ could converge or diverge.

D. None of the above.



The series is centered at $x=1$ and diverges when $x=-2$, which is 3 units from the center $\rightarrow$ The max ROC is 3.

We thus know the series must diverge if $x < -2$ or $x > 4$, but have no information what happens if $-2 < x \leq 4$, $x \neq 1$.

Since $f(2) = \sum_{k=0}^{\infty} a_k(2-1)^k = \sum_{k=0}^{\infty} a_k$, the series $\sum a_k$ could either converge or diverge.

IV. Of the following series:

i. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ ii. $\sum_{k=1}^{\infty} \frac{3^k}{k^{100}+2^k}$ iii. $\sum_{k=3}^{\infty} \cos\left(\frac{1}{k}\right)$

which must diverge by the divergence test?

A. i. only B. ii. only C. iii. only

D. i. and ii. E i. and iii. $\boxed{\text{F. ii. and iii.}}$

G. All of them H. None of them

The divergence test ensures that if $\lim_{k\to\infty} a_k \neq 0$, then $\sum_k^{\infty} a_k$ diverges.

Note: • $\lim_{k\to\infty} \frac{1}{k^2} = 0$ | • $\lim_{k\to\infty} \frac{3^k}{k^{100}+2^k} = \lim_{k\to\infty} \frac{3^k}{2^k\left(\frac{k^{100}}{2^k}+1\right)} = \lim_{k\to\infty} \frac{3}{2} = 1$ ($\frac{k^{100}}{2^k} \to 0$ by growth rates) | • $\lim_{k\to\infty} \cos\left(\frac{1}{k}\right) = \cos(0) = 1$.

V. Which of the following is the Taylor series centered at $x = 0$ for $f(x) = x\cos(x)$?

A. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$ B. $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ C. $\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+1)!}$

D. $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k)!}$ E. $\sum_{k=0}^{\infty} x^{2k+1}$ $\boxed{\text{F. None of the above.}}$

Since $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, $x\cos x = x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k)!}$$

2. **Short Answer** [16 pts]

   Answer each of the questions below. Each is worth 4 points.

I. Given that $f(-1) = 0$, $f'(-1) = 2$, $f''(-1) = 3$, and $f'''(-1) = 6$, write down the **second** degree Taylor polynomial of $f(x)$ centered at $x = -1$.

The second degree Taylor polynomial is $p_2(x) = a_0 + a_1(x+1) + a_2(x+1)^2$ where $a_k = \frac{f^{(k)}(-1)}{k!}$. Hence, $a_0 = 0$, $a_1 = 2$, $a_2 = \frac{3}{2!} = \frac{3}{2}$.

$\rightarrow \boxed{p_2(x) = 2(x+1) + \frac{3}{2}(x+1)^2}$

II. Give the general partial fraction decomposition for $\frac{4x+7}{x^4+x^2}$. DO NOT SOLVE FOR THE CONSTANTS!

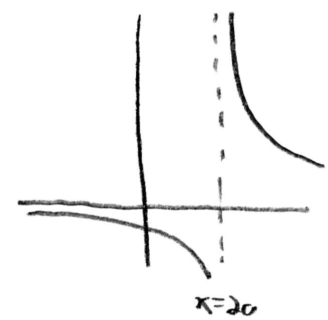
$$\frac{4x+7}{x^4+x^2} = \frac{4x+7}{\underbrace{x^2}_{\text{repeated linear}}\underbrace{(x^2+1)}_{\text{irreducible quadratic}}} = \boxed{\frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{x^2+1}}$$

III. For which real values of a is the integral $\int_0^1 \frac{1}{x-2a}\,dx$ improper? Explain your answer!

The integral is improper if $\frac{1}{x-2a}$ is unbounded on $[0,1]$. Since $\frac{1}{x-2a}$ has a vertical asymptote when $x = 2a$, the integral is improper if

$0 \le 2a \le 1$

$\boxed{0 \le a \le \frac{1}{2}}$



The graph of $y = \frac{1}{x-2a}$ has an asymptote at $x = 2a$. If this x-value lies in the interval of integration, the integral is improper.

IV. Write the expression below in summation notation:

$$1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots + \left(\frac{2}{3}\right)^{17}$$

$$\boxed{\sum_{k=0}^{17} \left(\frac{2}{3}\right)^k}$$

3. [16 pts] (Trigonometric substitution and improper integrals)

I. [11 pts] Use an appropriate trigonometric substitution to show that for $x > 0$:

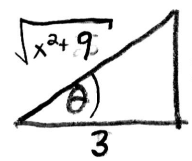
$$\int \frac{18}{(x^2+9)^{3/2}}\,dx = \frac{2x}{\sqrt{x^2+9}} + C$$

x^2+9 is of the form u^2+a^2, where $u = x$, $a = 3$.

$\rightarrow$ $u = a\tan\theta$

$x = 3\tan\theta$

$\rightarrow$ $dx = 3\sec^2\theta\, d\theta$.



$x = 3\tan\theta$

$\tan\theta = \frac{x}{3}$

So: $\displaystyle\int \frac{18}{(x^2+9)^{3/2}}\,dx = \int \frac{18}{(9\tan^2\theta+9)^{3/2}} \cdot 3\sec^2\theta\, d\theta$

Don't forget to convert dx!

$$= \int \frac{18}{(9\sec^2\theta)^{3/2}}\, 3\sec^2\theta\, d\theta$$

$$= \int \frac{54}{27\sec^3\theta}\, \sec^2\theta\, d\theta$$

DO NOT write: $\int \frac{1}{\sec\theta}\,d\theta = \ln|\sec\theta| + C$! What happens if you take $\frac{d}{d\theta}\ln|\sec\theta|$?

$$= 2\int \frac{1}{\sec\theta}\, d\theta$$

$$= 2\int \cos\theta\, d\theta$$

$$= 2\sin\theta + C = \frac{2x}{\sqrt{x^2+9}} + C$$

(from the picture)

II. [5 pts] Determine whether the improper integral:

$$\int_4^\infty \frac{18}{(x^2+9)^{3/2}}\,dx$$

converges or diverges. If it converges, give the value to which it converges. Make sure you use proper notation in your solution!

The improper integral converges if and only if $\displaystyle\lim_{b\to\infty}\int_4^b \frac{18}{(x^2+9)^{3/2}}\,dx$ exists.

$$\lim_{b\to\infty}\int_4^b \frac{18}{(x^2+9)^{3/2}}\,dx = \lim_{b\to\infty}\left[\frac{2x}{\sqrt{x^2+9}}\right]_4^b$$

$$= \lim_{b\to\infty}\left[\frac{2b}{\sqrt{b^2+9}} - \frac{8}{\sqrt{16+9}}\right]$$

$$= 2 - \frac{8}{5}$$

$$\frac{2b}{\sqrt{b^2+9}} = \frac{2b}{\sqrt{b^2}\sqrt{1+\frac{9}{b^2}}} = \frac{2b}{|b|}\frac{1}{\sqrt{1+9/b^2}} = 2\frac{1}{\sqrt{1+9/b^2}}$$

for $b > 0$.

$\boxed{\text{The improper integral converges to } \frac{2}{5}.}$

4. [18 pts] Suppose $f(x) = \sum_{k=1}^{\infty} \frac{k^2 x^{2k}}{4^{k-1}}$.

I. [8 pts] Find the radius of convergence for $f(x)$.

$$L(x) = \lim_{k\to\infty} \left| \frac{(k+1)^2 \; x^{2(k+1)}}{4^{(k+1)-1}} \cdot \frac{4^{k-1}}{k^2 x^{2k}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{(k+1)^2}{k^2} \cdot \frac{x^{2k+2}}{x^{2k}} \cdot \frac{4^{k-1}}{4^k} \right|$$

$$= \lim_{k\to\infty} \left| \left[\frac{k+1}{k}\right]^2 \cdot \frac{\cancel{x^{2k}}\, x^2}{\cancel{x^{2k}}} \cdot \frac{\cancel{4^{k-1}}}{\cancel{4^{k-1}}\,4} \right|$$

Make sure you understand the algebra here!

$$= \frac{|x|^2}{4} \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^2$$

$$= \frac{|x|^2}{4},$$

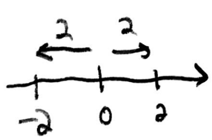
The series converges for all x for which $L(x) = \frac{|x|^2}{4} < 1$

$|x|^2 < 4$

$|x| < 2$

$\boxed{\text{The radius of convergence is } 2.}$

II. [4 pts] Explain whether the series for $f'(3)$ converges or diverges.



The series is centered at $x=0$ and has ROC 2. Differentiating does not change the ROC so the series for $f'(x)$ will converge for $-2 < x < 2$ and diverge if $x > 2$ or $x < -2$.

Hence, $\boxed{f'(3) \text{ diverges}}$.

III. [6 pts] Compute $\lim_{x\to 0} \frac{f(x) - x^2}{2x^4}$.

Note, by writing out the first several terms in the series for $f(x)$:

$$f(x) = x^2 + x^4 + \frac{9}{8}x^6 + \ldots$$

so

$$\lim_{x\to 0} \frac{f(x) - x^2}{2x^4} = \lim_{x\to 0} \frac{(\cancel{x^2} + x^4 + \frac{9}{8}x^6 + \ldots) - \cancel{x^2}}{2x^4}$$

$$= \lim_{x\to 0} \frac{\cancel{x^4}\left[1 + \frac{9}{8}x^2 + \ldots\right]}{2\cancel{x^4}}$$

$$= \boxed{\frac{1}{2}}$$

5. [30 pts] For $-\dfrac{1}{2} < x < \dfrac{1}{2}$, define the function $f(x)$ by the definite integral below[1]:

$$f(x) = \int_0^x \frac{8}{1-4t^2}\, dt.$$

I. For a Taylor series $g(t) = \sum_{k=0}^{\infty} a_k (t-c)^k$, the phrase "the sum of the first n nonzero terms" means the sum of the first n powers (in ascending order) of $(x-c)$ whose coefficients are nonzero.

A. [6 pts] Write out the first 3 nonzero terms in the Taylor series for $\dfrac{8}{1-4t^2}$ centered at $t = 0$.

$$\frac{1}{1-t} = 1 + t + t^2 + \ldots$$

$$\frac{1}{1-4t^2} = 1 + (4t^2) + (4t^2)^2 + \ldots$$

$$\frac{8}{1-4t^2} = 8\left[1 + 4t^2 + 16t^4 - \ldots\right]$$

$$= \boxed{8 + 32t^2 + 128t^4 - \ldots}$$

B. [4 pts] Let $p(t)$ denote the polynomial you found in part A. Calculate $\int_0^x p(t)\, dt$.

$p(t) = 8 - 32t^2 + 128t^4$ so $\int_0^x p(t)\, dt = \int_0^x (8 + 32t^2 + 128t^4)\, dt$

$$= \left[8t + \frac{32}{3}t^3 + \frac{128}{5}t^5\right]_0^x$$

$$= \boxed{8x + \frac{32}{3}x^3 + \frac{128}{5}x^5}$$

[1] So, for instance, $f(1/3)$ would be found by computing $\int_0^{1/3} \frac{8}{1-4t^2}\, dt$.

You do not need to know how to do Part I. in order to work the rest of this problem!

II. [8 pts] Compute $\int \frac{8}{1-4x^2}\,dx$.

Since $\frac{8}{1-4x^2} = \frac{8}{(1-2x)(1+2x)}$, we can use partial fraction decomposition:

$$\frac{8}{(1-2x)(1+2x)} = \frac{A}{1-2x} + \frac{B}{1+2x}$$

$$8 = A(1+2x) + B(1-2x)$$

$\underline{x=-\frac{1}{2}:}$ $8 = 2B \rightarrow \underline{B=4}$

$\underline{x=\frac{1}{2}:}$ $8 = 2A \rightarrow \underline{A=4}$.

So $\int \frac{8}{1-4x^2}\,dx = \int \left(\frac{4}{1-2x} + \frac{4}{1+2x}\right)dx$

$= \boxed{-2\ln|1-2x| + 2\ln|1+2x| + C}$

Make sure you understand why the coeff on $\ln|\ |$ are what they are!

III. [2 pts] Recall that $f(x) = \int_0^x \frac{8}{1-4t^2}\,dt$ for $-\frac{1}{2} < x < \frac{1}{2}$.

Deduce that $f(x) = 2\ln(1+2x) - 2\ln(1-2x)$.

$$f(x) = \int_0^x \frac{8}{1-4t^2}\,dt = \left[-2\ln|1-2t| + 2\ln|1+2t|\,\Big|_0^x\right]$$

$$= \left[-2\ln|1-2x| + 2\ln|1+2x|\right] - \left[-2\ln 1 + 2\ln 1\right]$$

Since $-\frac{1}{2} < x < \frac{1}{2}$, $\ln|1 \pm 2x| = \ln(1 \pm 2x)$ so:

$$f(x) = 2\ln(1+2x) - 2\ln(1-2x)$$

IV. [10 pts] Use the fact that:

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

and the usual rules for Taylor series of sums and compositions to write out the sum of the first 3 nonzero terms in the Taylor series centered at $x = 0$ for $f(x) = 2\ln(1+2x) - 2\ln(1-2x)$. **Simplify** your final answer!

From I, we expect we should write out powers up to x^5:

- $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \ldots$

- $\ln(1-2x) = -(2x) - \frac{1}{2}(2x)^2 - \frac{1}{3}(2x)^3 - \frac{1}{4}(2x)^4 - \frac{1}{5}(2x)^5 - \ldots$

  $= -2x - \frac{1}{2}(4x^2) - \frac{1}{3}(8x^3) - \frac{1}{4}(16x^4) - \frac{1}{5}(32x^5) - \ldots$

  $= -2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 - \frac{32}{5}x^5 - \ldots$

- $2\ln(1-2x) = -4x - 4x^2 - \frac{16}{3}x^3 - 8x^4 - \frac{64}{5}x^5 - \ldots$

- $\ln(1+2x) = \ln(1-(-2x))$

  $= -(-2x) - \frac{1}{2}(-2x)^2 - \frac{1}{3}(-2x)^3 - \frac{1}{4}(-2x)^4 - \frac{1}{5}(-2x)^5 - \ldots$

  $= 2x - \frac{1}{2}(4x^2) - \frac{1}{3}(-8x^3) - \frac{1}{4}(16x^4) - \frac{1}{5}(-32x^5) - \ldots$

  $= 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \frac{32}{5}x^5 - \ldots$

- $2\ln(1+2x) = 4x - 4x^2 + \frac{16}{3}x^3 - 8x^4 + \frac{64}{5}x^5 - \ldots$

Thus, $2\ln(1+2x) - 2\ln(1-2x) = 4x - \cancel{4x^2} + \frac{16}{3}x^3 - \cancel{8x^4} + \frac{64}{5}x^5 - \ldots$

$\qquad - \left(-4x - \cancel{4x^2} - \frac{16}{3}x^3 - \cancel{8x^4} - \frac{64}{5}x^5 - \ldots\right)$

$$= \boxed{8x + \frac{32}{3}x^3 + \frac{128}{5}x^5 + \ldots}$$

This matches I, B.