DISCLAIMER

General Information:

This is a midterm from a previous semester. This means:

- This midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to this midterm.
- This midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take this exam:

You should treat this midterm should be as the actual exam. This means:

- "Practice like you play." Schedule 55 uninterrupted minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and projects with this intent, and there will be a problem on your midterm that will require you to apply the material in an unfamiliar setting.

How to use the solutions:

DO NOT JUST READ THE SOLUTIONS!!!

The least important aspect of the solutions is learning the steps necessary to solve a specific problem. You should be looking for the concepts required to provide solutions. Content may not be recycled, but concepts will be!

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

Math 1172 Name: Solutions

Midterm 3 OSU Username (name.nn): ____________________

Autumn 2016 Lecturer: ____________________

Recitation Instructor: ____________________

Form A Recitation Time: ____________________

Instructions

- You have **55 minutes** to complete this exam. It consists of 6 problems on 12 pages including this cover sheet. Page 11 has possibly helpful formulas and may also be used for extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	16	
2	10	
3	24	
4	14	
5	18	
6	18	
Total	100	

1. **Multiple Choice** [16 pts]

   Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

I. Find a vector orthogonal to $\vec{u} = \hat{\jmath} + 3\hat{k}$ and $\vec{v} = 3\hat{\imath} + 2\hat{\jmath}$

A. $6\hat{\imath} - 9\hat{\jmath} - 3\hat{k}$

B. $-6\hat{\imath} - 9\hat{\jmath} - 3\hat{k}$

C. $-6\hat{\imath} + 9\hat{\jmath} + 3\hat{k}$

D. $-6\hat{\imath} - 9\hat{\jmath} + 3\hat{k}$

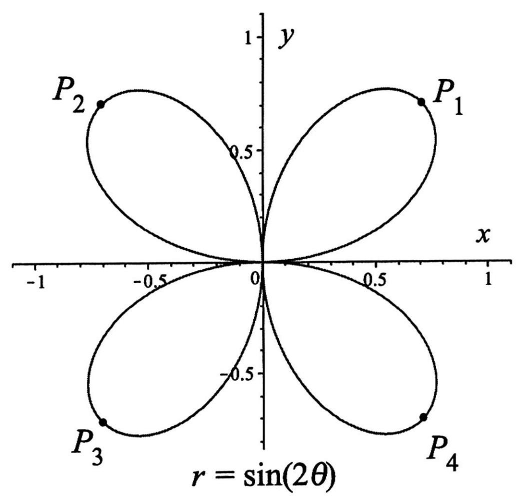
E. $6\hat{\imath} + 9\hat{\jmath} + 3\hat{k}$

$\boxed{\text{F. None of the above}}$

Such a vector must be parallel to $\vec{u} \times \vec{v} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 3 \\ 3 & 2 & 0 \end{bmatrix}$

$= -6\hat{\imath} + 9\hat{\jmath} - 3\hat{k}$

II. The curve C described by the polar equation $r = \sin(2\theta)$ is shown below:



Which of the following represents the point on the curve when $\theta = \dfrac{3\pi}{4}$?

A. P_1

B. P_2

C. P_3

$\boxed{\text{D. } P_4}$

E. More than one of these

F. None of the above

Also, if $r=-1$, $\theta = \frac{3\pi}{4}$

When $\theta = \frac{3\pi}{4}$, $r(\theta) = \sin \frac{3\pi}{2} = -1$.

- $x = r\cos\theta = -1 \cos\frac{3\pi}{4} = -1\left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$ → P_4 is the point!
- $y = r\sin\theta = -1 \sin\frac{3\pi}{4} = -1\left(\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2}$

III. The curve C is described by the polar equation $r = \dfrac{\theta^2 \sec(\theta)}{4}$.

Given that $\dfrac{dy}{d\theta} = \dfrac{\pi^2}{4}$ when $\theta = \pi$, find $\dfrac{dy}{dx}$ when $\theta = \pi$.

A. $\dfrac{dy}{dx} = 1$

B. $\boxed{\dfrac{dy}{dx} = \dfrac{\pi}{2}}$

C. $\dfrac{dy}{dx} = \dfrac{\pi^2}{2}$

D. $\dfrac{dy}{dx} = \dfrac{2}{\pi}$

E. $\dfrac{dy}{dx} = \dfrac{2}{\pi^2}$

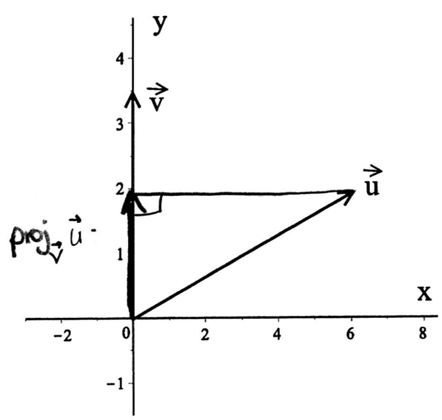
F. None of the above

$x = r\cos\theta = \left(\dfrac{\theta^2}{4}\sec\theta\right)\cos\theta = \dfrac{\theta^2}{4} \rightarrow \dfrac{dx}{d\theta} = \dfrac{1}{2}\theta.$

Thus, when $\theta = \pi$, $\dfrac{dx}{d\theta} = \dfrac{\pi}{2}$.

Since $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$, when $\theta = \dfrac{\pi}{2}$, $\dfrac{dy}{dx} = \dfrac{\pi^2/4}{\pi/2} = \boxed{\dfrac{\pi}{2}}$

IV. Using the vectors $\vec{u}$ and $\vec{v}$ shown below, determine the best choice for $proj_{\vec{v}}\vec{u}$.



A. $proj_{\vec{v}}\vec{u} = 3.5$

B. $proj_{\vec{v}}\vec{u} = 3.5\hat{\jmath}$

C. $proj_{\vec{v}}\vec{u} = 2$

D. $\boxed{proj_{\vec{v}}\vec{u} = 2\hat{\jmath}}$

E. $proj_{\vec{v}}\vec{u} = \vec{u} \times \vec{v}$

F. $proj_{\vec{v}}\vec{u} = \dfrac{1}{2}\vec{u}$

2. **Multianswer** [10 pts]

Directions: A perfect answer for this question is worth 10 points. You will be penalized 2 points for each incorrect response. Thus, the possible grades on each problem are 0, 2, 4, 6, 8, or 10. You cannot score below a 0 for this problem.

Problem: Suppose $\vec{u}$ and $\vec{v}$ are nonzero vectors in the xyz-plane. The symbol $\cdot$ refers to the vector dot product, and $\times$ refers to the vector cross product. Indicate whether the following are *vectors*, *scalars*, or *undefined* in the space provided.

A. $\vec{u} \cdot \vec{v}$ scalar

B. $(\vec{u} + \vec{v})\,\vec{v}$ undefined

C. $proj_{\vec{v}}\,\vec{u}$ vector

D. $\underbrace{(\vec{u} \cdot \vec{v})}_{\text{scalar!}} \times \vec{u}$ undefined

E. $(\vec{u} \times \vec{v}) \times \vec{u}$ vector

F. $\dfrac{\vec{u}}{\vec{v}}$ undefined

G. $(\underbrace{scal_{\vec{u}}\,\vec{v}}_{\text{scalar}})\,\vec{v}$ vector

H. $(proj_{\vec{v}}\,\vec{u}) \cdot \vec{v}$ scalar

<u>Facts</u>:
- When defined, dot products are scalars
  cross products are vectors
- $scal_{\vec{v}}\,\vec{u}$ is a scalar
- $proj_{\vec{v}}\,\vec{u}$ is a vector
- dot products and cross products between vectors are defined.
- "products" of vectors like $\vec{u}\vec{v}$ or $\frac{\vec{u}}{\vec{v}}$ are not defined!
- products of scalars and vectors are defined.

3. **Short Answer** [24 pts]

Answer each of the following questions and justify your responses unless otherwise requested. Each question is worth 4 points.

I. Determine whether the following statement is True or False. If the statement is true, briefly explain your answer. If it is false, provide a counterexample.

If $\vec{u}$ and $\vec{v}$ are nonzero vectors and $proj_{\vec{v}}\vec{u} = \vec{0}$, then $\vec{u}$ and $\vec{v}$ must be orthogonal.

True; $proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$. This is $\vec{0}$ if either $\vec{u}\cdot\vec{v} = 0$ or $\vec{v} = 0$.

Since $\vec{v}$ is nonzero, $\vec{u}\cdot\vec{v}$ must be 0. Thus, $\vec{u}$ and $\vec{v}$ are orthogonal.

II. CIRCLE the correct response to each question. NO justification is necessary!

If $\vec{u}$ and $\vec{v}$ are __________, then $\vec{u}\cdot\vec{v} = 0$.

PARALLEL $\qquad$ $\boxed{\text{PERPENDICULAR}}$

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$
$= 0$ if $\cos\theta = 0$
$\Rightarrow \theta = \frac{\pi}{2}$

If $\vec{u}$ and $\vec{v}$ are __________, then $\vec{u}\times\vec{v} = \vec{0}$.

$\boxed{\text{PARALLEL}}$ $\qquad$ PERPENDICULAR

$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$
$= 0$ if $\theta = 0, \pi$.

III. Find all values of a such that the curve C described by the polar equation $r(\theta) = 2a - 3\cos\theta$ passes through the origin in the xy-plane.

The curve passes through the origin if there is a θ-value for which

$r(\theta) = 0 \rightarrow \quad 2a - 3\cos\theta = 0$

$$\cos\theta = \frac{2}{3}a.$$

The range of $\cos\theta$ is $[-1, 1]$, so such a θ-value exists if

$$-1 \leq \frac{2}{3}a \leq 1$$

$$\boxed{-\frac{3}{2} \leq a \leq \frac{3}{2}}$$

* Try graphing this (on Desmos, for example) for $a = -6, -3, -\frac{3}{2}, -1, -\frac{1}{2}, 0$ to see what's happening visually.

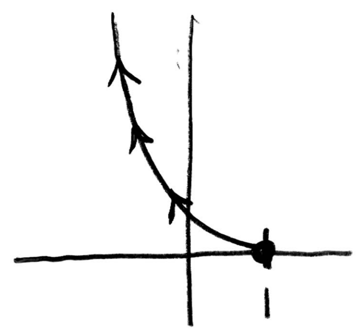
IV. The curve C is described parametrically by:

$$\begin{cases} x(t) &= 1 - t^2 \\ y(t) &= t^4 \end{cases}, \quad t \geq 0$$

Eliminate the parameter to find a description in terms of x and y only. Sketch the curve and indicate the positive orientation on your sketch.

$x = 1 - t^2 \rightarrow t^2 = 1 - x$

$y = t^4 = (t^2)^2 = (1-x)^2$



C is thus part of the parabola $y = (1-x^2)$

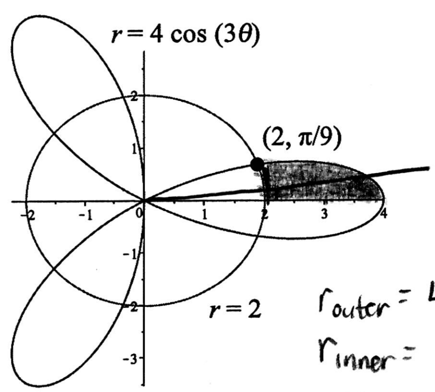
Note when $t = 0$, $x = 1$, $y = 0$

When t increases, x decreases and y increases,

so we obtain the left half of the parabola.

V. The polar curves $r = 4\cos(3\theta)$ and $r = 2$, which intersect at $(r, \theta) = (2, \pi/9)$ in QI.

Set up, but DO NOT EVALUATE, an integral or sum of integrals in terms of θ that expresses the *total* area inside the curve $r = 4\cos(3\theta)$ but outside of the circle $r = 2$.



$r_{outer} = 4\cos 3\theta$

$r_{inner} = 2$

The total area is 6 times the shaded area.

$$A = 6\left[\int_{\theta=0}^{\theta=\pi/9} \frac{1}{2}\left(r_{outer}^2 - r_{inner}^2\right) d\theta\right].$$

$$= \boxed{3\int_0^{\pi/9} \left[16\cos^2 3\theta - 4\right] d\theta}$$

VI. Suppose that $\hat{v}$ is a unit vector and $\vec{u}$ is a nonzero vector. Show that $scal_{\hat{v}}\,\vec{u} = \vec{u} \cdot \hat{v}$.

By definition, $|\hat{v}| = 1$ and $scal_{\hat{v}}\,\vec{u} = \dfrac{\vec{u} \cdot \hat{v}}{|\hat{v}|}$

$$= \frac{\vec{u} \cdot \hat{v}}{1}$$

$$= \vec{u} \cdot \hat{v}$$

4. [14 pts] A curve is described parametrically by:

$$\begin{cases} x(t) &= t^2 + 2t + 2 \\ y(t) &= 4t + 1 \end{cases}$$

for all t where $x(t)$ and $y(t)$ are well-defined.

I. Find $\frac{dy}{dx}$ in terms of t. Simplify your final answer!

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4}{2t+2} = \boxed{\frac{2}{t+1}}$$

II. Give the Cartesian equation(s) for all vertical tangent lines to the curve or state that there are none.

There are vertical tangent lines when $\frac{dy}{dx}$ is unbounded

$\rightarrow$ $t+1=0$ $\rightarrow$ $\underline{t=-1}$.

The equations of the vertical tangent line is $x = \underline{x(-1) = (-1)^2 + 2(-1) + 2}$

$\boxed{x = 1}$

III. Find the Cartesian equation(s) for all tangent lines to the curve with slope 2.

- The tangent line has slope 2 when $\frac{dy}{dx} = \frac{2}{t+1} = 2$

$\cancel{2} = 2t + \cancel{2}$

$\underline{t = 0}$.

- The point of tangency is thus $x(0) = 0^2 + 2(0) + 2 \rightarrow x = 2$

$y(0) = 4(0) + 1 \rightarrow y = 1$.

- The tangent line is thus:

$y - y(0) = m[x - x(0)]$

$y - 1 = 2[x - 2]$

$\boxed{y = 2x - 3}$

5. [18 pts] A curve C in space is given described by the vector-valued function:

$$\vec{r}(t) = \langle 2t^2 - 1,\, t^2 + 5,\, 2t \rangle .$$

I. Find the unit tangent vector to the curve.

Not a unit vector!

A vector parallel to $\hat{T}(t)$ is $\vec{r}\,'(t) = \langle 4t, 2t, 2 \rangle$

So, $|\vec{r}\,'(t)| = \sqrt{(4t)^2 + (2t)^2 + (2)^2}$

$= \sqrt{16t^2 + 4t^2 + 4}$

$= \sqrt{4 + 20t^2} = \sqrt{4(1+5t^2)} = 2\sqrt{1+5t^2}$

Hence, $\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\langle 4t, 2t, 2 \rangle}{2\sqrt{1+5t^2}} = \boxed{\dfrac{1}{\sqrt{1+5t^2}} \langle 2t, t, 1 \rangle}$

II. Find all values of t where $\vec{r}(t)$ and $\vec{r}\,'(t)$ are orthogonal.

We want t so $\vec{r}(t) = \langle 2t^2 - 1, t^2 + 5, 2t \rangle$ and $\vec{r}\,'(t) = \langle 4t, 2t, 2 \rangle$ are orthogonal $\rightarrow$ $\langle 2t^2-1, t^2+5, 2t \rangle \cdot \langle 4t, 2t, 2 \rangle = 0$

$4t(2t^2 - 1) + 2t(t^2+5) + 2(2t) = 0$

$8t^3 - \cancel{4t} + 2t^3 + 10t + \cancel{4t} = 0$

$10t^3 + 10t = 0$

$10t(t^2+1) = 0 \Rightarrow \boxed{t = 0}$

III. Give a parametric description of the tangent line to C when $t = 2$.

- A parametric description of a line is:

  $\vec{\ell}(t) = \vec{v}t + \vec{P_0}$

  where $\vec{v}$ is a vector parallel to the line and P_0 is a point on the line.

- $\vec{r}\,'(2)$ is parallel to the tangent line, so $\vec{v} = \vec{r}\,'(2) = \langle 8, 4, 2 \rangle$
- $\vec{r}(2)$ is the point of tangency so $\vec{P_0} = \vec{r}(2) = \langle 7, 9, 4 \rangle$

Hence, $\vec{\ell}(t) = \langle 8, 4, 2 \rangle t + \langle 7, 9, 4 \rangle$

$= \boxed{\langle 8t+7,\, 4t+9,\, 2t+4 \rangle}$

6. [18 pts] The trajectory of a particle in space is given described by the vector-valued function:

$$\vec{r}(t) = \left\langle \sin(t^3), \frac{5}{3}\cos(t^3), \frac{4}{3}\sin(t^3) \right\rangle, \quad 0 \le t \le 2.$$

I. Show that the speed associated with the trajectory is $|\vec{v}(t)| = 5t^2$.

The velocity $\vec{v}(t)$ is $\vec{r}\,'(t) = \langle 3t^2\cos(t^3), -5t^2\sin(t^3), 4t^2\cos(t^3) \rangle$

The speed is $|\vec{v}(t)| = \sqrt{[3t^2\cos(t^3)]^2 + [-5t^2\sin(t^3)]^2 + [4t^2\cos(t^3)]^2}$

$$= \sqrt{9t^4\cos^2(t^3) + 25t^4\sin^2(t^3) + 16t^4\cos^2(t^3)}$$

$$= \sqrt{25t^4\underbrace{[\cos^2(t^3) + \sin^2(t^3)]}_{1}}$$

$$= \sqrt{25t^4} = \boxed{5t^2}$$

II. Explain whether the curve is parameterized by arclength. If it is not, find a description that uses arclength as a parameter. Make sure to give the bounds of the arclength parameter in this case.

- A curve is parameterized by arclength if and only if $|\vec{r}\,'(t)| = 1$ for all t. Here, $|r'(t)| = 5t^2$, so the curve is not parameterized by arclength.

- To parameterize by arclength:

$$s = \int_0^t |\vec{r}\,'(\tau)|\, d\tau$$

$$s = \int_0^t 5\tau^2 d\tau$$

$$s = \frac{5}{3}t^3$$

Substituting $t^3 = \frac{3}{5}s$ is easier than solving for t and substituting!

So $t^3 = \frac{3}{5}s$. and $\boxed{\vec{r}(s) = \left\langle \sin\left(\frac{3}{5}s\right), \frac{5}{3}\cos\left(\frac{3}{5}s\right), \frac{4}{3}\sin\left(\frac{3}{5}s\right) \right\rangle}$

Note if $0 \le t \le 2$, we have $0 \le t^3 \le 8$

$$0 \le \frac{3}{5}s \le 8$$

$$\boxed{0 \le s \le \frac{40}{3}}$$