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General Information:

This is a midterm from a previous semester. This means:

- This midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to this midterm.
- This midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take this exam:

You should treat this midterm should be as the actual exam. This means:

- "Practice like you play." Schedule 55 uninterrupted minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and projects with this intent, and there will be a problem on your midterm that will require you to apply the material in an unfamiliar setting.

How to use the solutions:

DO NOT JUST READ THE SOLUTIONS!!!

The least important aspect of the solutions is learning the steps necessary to solve a specific problem. You should be looking for the concepts required to provide solutions. Content may not be recycled, but concepts will be!

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

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Math 1172

Midterm 2

Spring 2016

Form A

Name: Solutions

OSU Username (name.nn): ____________

Lecturer: ____________

Recitation Instructor: ____________

Recitation Time: ____________

Instructions

- You have **55 minutes** to complete this exam. It consists of 6 problems on 10 pages including this cover sheet. Page 9 has possibly helpful formulas and pages 9 and 10 may also be used for extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	12	
2	10	
3	18	
4	16	
5	14	
6	30	
Total	100	

1. **Multiple Choice** [12 pts]

Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

I. Given that $\sum_{k=1}^{\infty} \frac{12}{k^2+2k} = 9$, find $\sum_{k=2}^{\infty} \frac{12}{k^2+2k}$.

A. 4 $\quad$ $\boxed{\text{B. 5}}$ $\quad$ C. 9 $\quad$ D. 13 $\quad$ E. None of the above

$$\sum_{k=1}^{\infty} \frac{12}{k^2+2k} = \frac{12}{(1)^2+2(1)} + \sum_{k=2}^{\infty} \frac{12}{k^2+2k}$$

$$9 = 4 + \sum_{k=2}^{\infty} \frac{12}{k^2+2k} \rightarrow \sum_{k=2}^{\infty} \frac{12}{k^2+2k} = 5$$

II. Suppose $a_n = \frac{(-1)^n}{n}$ for all $n \geq 1$. Then, the sequence $\{a_n\}_{n\geq 1}$ is:

$\boxed{\text{A. Bounded only}}$ $\quad$ B. Monotonic only

C. Both bounded and monotonic $\quad$ D. Neither bounded nor monotonic.

$\frac{(-1)^n}{n}$ alternates in sign so it is NOT always increasing nor decreasing!

III. Suppose $\sum_{k=1}^{\infty} a_k$ diverges, and $M > 1$ is an integer. Then, $\sum_{k=M}^{\infty} a_k$:

A. Must converge. $\quad$ B. Could converge or diverge.

$\boxed{\text{C. Must diverge.}}$ $\quad$ D. Converges only if M is sufficiently large.

Whether a series converges or diverges does NOT depend on the lower index (first finitely many terms).

2. **Multiselect** [10 pts] Circle *all* of the responses that MUST be true. Note that there may be more than one correct response or even no correct responses! A perfect answer is worth 10 points, and you will be penalized 2 points for each incorrect selection. You cannot score below a 0 for this problem.

The first four nonzero terms in the Taylor series for a certain function $f(x)$ centered at $x = 0$ are:

$$f(x) = 3 + 2x^2 - 5x^3 + x^5 + \dots$$

Suppose it is known that:

- The series for $f(1)$ converges. $\leftarrow$ $x=1$ is 1 unit from the center $\Rightarrow$ ROC ≥ 1
- The series for $f(-2)$ diverges. $\leftarrow$ $x=-2$ is 2 units from the center $\Rightarrow$ ROC ≤ 2.

CIRCLE *all* of the following statements that *MUST* be true.

A. $f(0) = 3$

B. $f'(0) = 2$

C. $f'''(0) = -5$

D. $f(2x) = 3 + 8x^2 - 40x^3 + 32x^5 + \dots$

E. The series must converge for $x = 0$.

F. The series must converge for $x = 1/2$.

G. The radius of convergence for the series for $f(x)$ is at most 2.

H. The radius of convergence for the series for $f'(x)$ is at least 1.

A. TRUE; $f(0) = 3 + 2(0)^2 + \dots = 0$

B. FALSE; $f'(x) = 4x - 15x^2 + 5x^4 + \dots \rightarrow f'(0) = 0$.

C. FALSE; $f''(x) = 4 - 30x + 20x^3 + \dots$

$f'''(x) = -30 + 60x^2 + \dots \rightarrow f'''(0) = -30$.

D. TRUE; $f(2x) = 3 + 2(2x)^2 - 5(2x)^3 + (2x)^5 + \dots = 3 + 8x^2 - 40x^3 + 32x^5 + \dots$

E. TRUE; Taylor series always converge at their centers!

F. TRUE; the series converges at $x=1$, so the ROC is at least 1.

G. TRUE; the series diverges at $x=-2$, so the ROC cannot be larger than 2.

H. TRUE; differentiation does not change the ROC of a series!

3. **Short Answer** [18 pts]

Answer the following questions. Each question is worth 3 points. You do not need to justify your answer. There is no partial credit and no penalty for guessing.

I. Suppose that $\{a_n\}_{n\geq 1}$ is a sequence for which $s_n = \frac{n^3}{6-n^2}$, where $s_n = \sum_{k=1}^{n} a_k$ for $n \geq 1$.

A. $a_3 =$ <u>-13</u>.

$a_3 = s_3 - s_2 = \frac{(3)^3}{6-(3)^2} - \frac{(2)^3}{6-(2)^2} = -\frac{27}{3} - \frac{8}{2}$

$= -13.$

B. $a_1 + a_2 + a_3 =$ <u>-9</u>.

$a_1 + a_2 + a_3 = s_3$: But we found above that $s_3 = -9$

C. Determine whether $\lim_{n\to\infty} s_n$ exists. If it does, give its value.

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n^3}{6-n^2} = \infty$. Thus, $\boxed{\lim_{n\to\infty} s_n \text{ DNE}}$

D. Determine whether $\sum_{k=1}^{\infty} a_k$ converges or diverges. If it converges, give the value to which it converges.

Since $\lim_{n\to\infty} s_n$ DNE, $\boxed{\sum_{k=1}^{\infty} a_k \text{ diverges}}$

II. Give the general partial fraction decomposition for $\frac{2x+1}{x^3+x^2}$. DO NOT SOLVE FOR THE CONSTANTS!

$\frac{2x+1}{x^3+x^2} = \frac{2x+1}{x^2(x+1)} = \boxed{\frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+1}}$

III. For which real values of b does the series $\sum_{k=1}^{\infty} \left(\frac{b}{2}\right)^k$ converge?

This is geometric, so $\left|\frac{b}{2}\right| < 1 \Rightarrow \boxed{|b| < 2}$

4. [16 pts] Given that $\dfrac{5}{(2x-1)(x^2+1)} = \dfrac{4}{2x-1} - \dfrac{2x+1}{x^2+1}$, determine if the improper integral:

$$\int_1^{\infty} \frac{5}{(2x-1)(x^2+1)}\,dx$$

converges or diverges. If it converges, give the value to which it converges and **simplify** your final answer. Make sure you use proper notation in your solution!

Note: $\displaystyle\int \frac{5}{(2x-1)(x^2+1)}\,dx = \int\left[\frac{4}{2x-1} - \frac{2x+1}{x^2+1}\right]dx$

$$= \int\left[\frac{4}{2x-1} - \frac{2x}{x^2+1} - \frac{1}{x^2+1}\right]dx$$

$$= 2\ln|2x-1| - \ln|x^2+1| - \arctan x + C$$

So, $\displaystyle\int_1^{\infty} \frac{5}{(2x-1)(x^2+1)}\,dx = \lim_{b\to\infty}\left[2\ln|2x-1| - \ln|x^2+1| - \arctan x\,\Big|_1^b\right.$

$\uparrow$ $\infty-\infty$ is NOT defined! Combine ln( ) terms!

$$= \lim_{b\to\infty}\left[\ln(2x-1)^2 - \ln(x^2+1) - \arctan x\,\Big|_1^b\right.$$

$$= \lim_{b\to\infty}\left[\ln\frac{(2x-1)^2}{x^2+1} - \arctan x\,\Big|_1^b\right.$$

$$= \lim_{b\to\infty}\left\{\left[\ln\frac{(2b-1)^2}{b^2+1} - \arctan b\right] - \left[\ln\frac{1}{2} - \arctan 1\right]\right\}$$

$$= \ln 4 - \frac{\pi}{2} - \ln\frac{1}{2} + \frac{\pi}{4}$$

$$= \boxed{\ln 8 - \frac{\pi}{4}}$$

5. [14 pts] Determine if the following series converge or diverge and **justify** your response!

I. $\displaystyle\sum_{k=1}^{\infty} \cos\left(\frac{k+1}{k}\right)$.

Try divergence test: $\displaystyle\lim_{k\to\infty} \frac{k+1}{k} = 1$

so $\displaystyle\lim_{k\to\infty} \cos\left(\frac{k+1}{k}\right) = \cos 1$

Since $\cos 1 \neq 0$, the series diverges by divergence test!

II. $\displaystyle\sum_{k=2}^{\infty} 2^{3-2k}$.

Note: $2^{3-2k} = 2^3 \cdot 2^{-2k} = \dfrac{8}{2^{2k}} = \dfrac{8}{(2^2)^k} = \dfrac{8}{4^k} = 8\left(\frac{1}{4}\right)^k$.

So, $\displaystyle\sum_{k=2}^{\infty} 2^{3-2k} = \sum_{k=2}^{\infty} 8\left(\frac{1}{4}\right)^k$.

This is a geometric series with $r = \frac{1}{4} < 1$, so it converges!

6. [30 pts] **Taylor Series**

I. [15 pts] Find the first four nonzero terms in the Taylor series centered at $x = 0$ for the function

$$f(x) = x + 6e^{x^2} - 3x^3 \sin x.$$

Simplify your final answer completely!

Hint: Using the definition to do this is *very* painful!

Use common Taylor series:

- $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots$

$e^{x^2} = 1 + (x^2) + \frac{1}{2}(x^2)^2 + \frac{1}{6}(x^2)^3 + \frac{1}{24}(x^2)^4$

$6e^{x^2} = 6 + 6x^2 + 3x^4 + x^6 + \frac{1}{4}x^8 + \ldots$

- $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \ldots$

$3x^3 \sin x = 3x^4 - \frac{1}{2}x^6 + \frac{1}{40}x^8 - \ldots$

So: $x + 6e^{x^2} - 3x^3 \sin x = x + \left[6 + 6x^2 + 3x^4 + x^6 + \frac{1}{4}x^8 + \ldots\right]$

$- \left[3x^4 - \frac{1}{2}x^6 + \frac{1}{40}x^8 - \ldots\right]$

No need to include!

$= \boxed{6 + x + 6x^2 + \frac{1}{2}x^6 + \ldots}$

II. For the power series:

$$f(x) = \sum_{k=1}^{\infty} \frac{4^k (x+2)^{2k}}{k}$$

A. [3 pts] State the center of the series.

The power series $\sum a_k (x-c)^k$ is centered at $x=c$, so this one is centered at $\boxed{x=-2}$

B. [12 pts] Find the radius of convergence of the power series.

Use the Ratio Test:

$$L(x) = \lim_{k\to\infty} \left| \frac{4^{k+1} (x+2)^{2(k+1)}}{k+1} \cdot \frac{k}{4^k (x+2)^{2k}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{4^{k+1}}{4^k} \cdot \frac{(x+2)^{2k+2}}{(x+2)^{2k}} \cdot \frac{k}{k+1} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\cancel{4^k} 4}{\cancel{4^k}} \cdot \frac{\cancel{(x+2)^{2k}} (x+2)^2}{\cancel{(x+2)^{2k}}} \cdot \frac{k}{k+1} \right|$$

$$= 4|x+2|^2 \lim_{k\to\infty} \frac{k}{k+1}$$

$$\underline{L(x) = 4|x+2|^2}$$

The ratio test ensures the series will converge for all values of x that make $L(x) < 1$:

$$L(x) = 4|x+2|^2 < 1$$

$$|x+2|^2 < \frac{1}{4}$$

$$|x+2| < \frac{1}{2}.$$

$\boxed{\text{The radius of convergence is } \frac{1}{2}.}$