- - - - - - - - - - - - - - - - - - **DISCLAIMER** - - - - - - - - - - - - - - - - - -

General Information:

This is a midterm from a previous semester. This means:

- This midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to this midterm.
- This midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take this exam:

You should treat this midterm should be as the actual exam. This means:

- "Practice like you play." Schedule 55 uninterrupted minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and projects with this intent, and there will be a problem on your midterm that will require you to apply the material in an unfamiliar setting.

How to use the solutions:

DO NOT JUST READ THE SOLUTIONS!!!

The least important aspect of the solutions is learning the steps necessary to solve a specific problem. You should be looking for the concepts required to provide solutions. Content may not be recycled, but concepts will be!

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

- -

Math 1172 Name: Solutions

Midterm 3 OSU Username (name.nn): __________

Spring 2016 Lecturer: __________

Recitation Instructor: __________

Form A Recitation Time: __________

Instructions

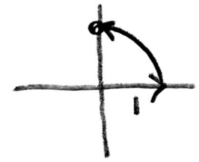
- You have **55 minutes** to complete this exam. It consists of 6 problems on 12 pages including this cover sheet. Page 11 has possibly helpful formulas and may also be used for extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	12	
2	18	
3	15	
4	20	
5	25	
6	10	
Total	100	

1. **Multiple Choice** [12 pts]

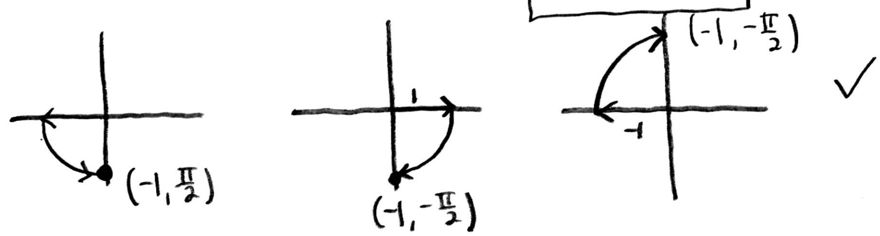
Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

I. A point in the xy-plane is described by the *polar coordinates* $(r, \theta) = \left(1, \frac{\pi}{2}\right)$.



Which of the following gives an alternate description of the point in polar coordinates?

A. $\left(-1, \frac{\pi}{2}\right)$ B. $\left(1, -\frac{\pi}{2}\right)$ $\boxed{\text{C. } \left(-1, -\frac{\pi}{2}\right)}$ D. None of the above



II. The level curves for the function $f(x, y) = 2y - 3x$ are:

A. Parabolas B. Ellipses C. Hyperbolas $\boxed{\text{D. None of the above}}$

Level curves are curves in the xy-plane where $2y - 3x = c$
These are lines!

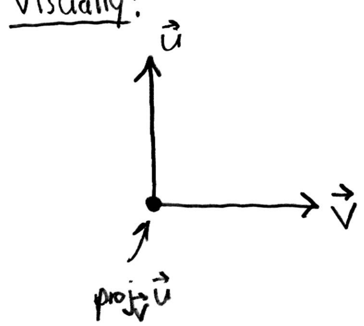
III. Suppose $\vec{u}$ and $\vec{v}$ are nonzero vectors and $\vec{u}$ is perpendicular to $\vec{v}$. Then:

$\boxed{\text{A. } proj_{\vec{v}}\,\vec{u} = \vec{0}}$ B. $proj_{\vec{v}}\,\vec{u} = \vec{u}$ C. $proj_{\vec{v}}\,\vec{u} = \vec{v}$

D. $proj_{\vec{v}}\,\vec{u}$ cannot be computed unless we know what the vectors $\vec{u}$ and $\vec{v}$ are.

E. None of the Above

<u>Visually:</u>



<u>Computationally</u>: $proj_{\vec{v}}\,\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$

$= \vec{0}$ since $\vec{u}\cdot\vec{v} = 0$!

2. **Multiselect** [18 pts]

Directions: Each problem below is worth 9 points. Circle *all* of the responses that MUST be true for each problem below. Note that there may be more than one correct response or even no correct responses!

A perfect answer for each part is worth 9 points. If you circle an incorrect choice, you will be penalized 3 points. If you do not circle a correct choice, you will be penalized 3 points. However, you cannot score below a 0 for either part of this problem. Thus, the possible scores for each part are 0, 3, 6, or 9 points.

I. [9 pts] Suppose $\vec{u}$ and $\vec{v}$ are constant nonzero three dimensional vectors.

CIRCLE *all* of the following quantities that are **vectors**:

I. $scal_{\vec{v}}\,\vec{u}$ $\qquad$ $\boxed{\text{II. } proj_{\vec{v}}\,\vec{u}}$ $\qquad$ III. $\vec{u}\cdot\vec{v}$ $\qquad$ $\boxed{\text{IV. } \vec{u}\times\vec{v}}$

$\boxed{\text{V. } \vec{u}+3\vec{v}}$ $\qquad$ VI. $\vec{u}\cdot(2\vec{v}+3\vec{u})$ $\qquad$ $\boxed{\text{VII. } \frac{d\vec{u}}{dt}}$ $\qquad$ $\boxed{\text{VIII. } (\vec{u}\cdot\vec{v})\,\vec{v}}$

- You need to know what objects are vectors and scalars and what the different operations produce!
- This question did <u>NOT</u> go as well as it should have collectively!

- Adding / subtracting vectors give vectors
- Dot products produce scalars
- Cross products produce vectors.
- Differentiating / Integrating vectors give vectors!

II. [9 pts] Suppose the curve C is defined parametrically by:

$$\begin{cases} x(t) &= 2t \\ y(t) &= t^3 - 3t \end{cases}, \quad t \geq 0$$

CIRCLE *all* of the following statements that MUST be true.

A. $\dfrac{dy}{dx}$ decreases as t increases.

B. The curve has no vertical tangent lines.

C. The point $(2, -2)$ is on the curve.

D. The curve has no horizontal tangent lines.

E. $\dfrac{dy}{dx} = -\dfrac{3}{2}$ when $x = 0$.

F. There is a time $t \geq 0$ when $\dfrac{dy}{dx} = 1$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2}$$

A. FALSE; clearly (since $t \geq 0$ only), $\frac{dy}{dx}$ increases as t increases!

B. TRUE; $\frac{dy}{dx}$ is finite for all finite times!

C. TRUE; When $x=2$, we find $x(t) = 2t = 2 \rightarrow t=1$

When $t=1$, $y(1) = (1)^3 - 3(1) = -2$ ✓

D. FALSE; $\frac{dy}{dx} = 0$ when $3t^2-3=0 \rightarrow t = +1$ (since we consider only $t \geq 0$, disregard $t=-1$)

E. TRUE; when $x=0$, $2t=0 \rightarrow t=0$.

$$\left.\frac{dy}{dx}\right|_{t=0} = \frac{3(0)^2-3}{2} = -\frac{3}{2}.$$

F. TRUE; We must look for $t \geq 0$ such that

$$\frac{dy}{dx} = \frac{3t^2-3}{2} = 1$$

$$3t^2-3 = 2$$

$$3t^2 = 5 \leftarrow \text{This has a solution!}$$

3. [15 pts] Consider the function $z = f(x, y) = \dfrac{x^2 + 2xy}{4x^2 - y^2}$.

I. Find $a > 0$ so that $(x, y) = (1, a)$ is *not* in the domain of $f(x, y)$.

$(1, a)$ is not in the domain if

$$4(1)^2 - a^2 = 0$$

$$a^2 = 4 \rightarrow a = \pm 2 \rightarrow \boxed{a = 2}.$$

II. Give the equation of the yz-trace of the surface.

For the yz-trace, set $x = 0$:

$$z = \frac{0^2 + 2(0)y}{4(0)^2 - y^2} \rightarrow \boxed{z = 0, \quad y \neq 0}$$

III. Determine whether $\lim_{(x,y)\to(0,0)} f(x, y)$ exists. Clearly justify your response!

Two-Path Test:

Along $x = 0$, we just showed $f(0, y) = 0$ for all $y \neq 0$

so $f(0, y) \rightarrow 0$ as $(x, y) \rightarrow (0, 0)$.

Along $y = 0$: $f(x, 0) = \dfrac{x^2 + 2x(0)}{4x^2 - 0^2} = \dfrac{1}{4}$, for all $x \neq 0$.

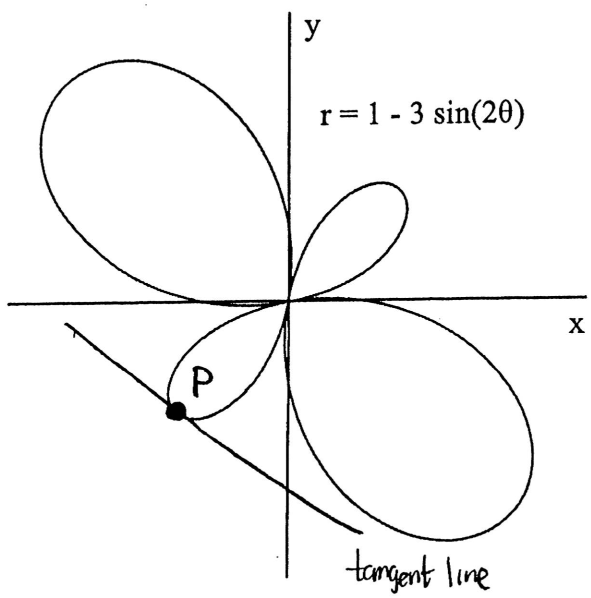
so $f(x, 0) \rightarrow \frac{1}{4}$ as $(x, y) \rightarrow (0, 0)$.

Since $f(x,y)$ tends to different values along different paths, $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

Or, you can note $f(x, mx) = \dfrac{x^2 + 2x(mx)}{4x^2 - (mx)^2} = \dfrac{1 + 2m}{4 - m^2}$, $x \neq 0$.

So, $f(x, mx) \rightarrow \dfrac{1 + 2m}{4 - m^2}$ as $(x, y) \rightarrow (0, 0)$ along $y = mx$. Since this value depends on the line we follow (i.e. the value of m), the limit DNE.

3. [20 pts] The curve described by the polar equation $r = 1 - 3\sin(2\theta)$ is shown below:



Note $r < 0$ when $\theta = \frac{\pi}{4}$!

I. On the figure plot the point on the curve when $\theta = \dfrac{\pi}{4}$. Label it P.

II. On the figure, draw the tangent line to the curve at P.

III. Find the Cartesian coordinates (x, y) of the point on the curve when $\theta = \dfrac{\pi}{4}$.

When $\theta = \frac{\pi}{4}$: • $r = 1 - 3\sin\left(\frac{\pi}{2}\right) = \underline{-2}$

• $x = r\cos\theta = -2\cos\frac{\pi}{4}$

$\rightarrow \boxed{x\left(\frac{\pi}{4}\right) = -\sqrt{2}}$

• $y = r\sin\theta = -2\sin\frac{\pi}{4}$

$\rightarrow \boxed{y\left(\frac{\pi}{4}\right) = -\sqrt{2}}$

* If you were confused by a negative r value in I, note that after doing III, you should be able to plot P correctly!
If you didn't compute r at all in I, **BE CAREFUL**! $(-r, \theta)$ is very different from $(+r, \theta)$!

IV. Find $\frac{dx}{d\theta}$ when $\theta = \frac{\pi}{4}$.

$$x = r\cos\theta = (1-3\sin 2\theta)\cos\theta.$$

$$\frac{dx}{d\theta} = -6\cos 2\theta\cos\theta - (1-3\sin 2\theta)\sin\theta$$

$$\left.\frac{dx}{d\theta}\right|_{\theta=\frac{\pi}{4}} = -6\overset{0}{\cancel{\cos\tfrac{\pi}{2}}}\cos\tfrac{\pi}{4} - (1-3\sin\tfrac{\pi}{2})\sin\tfrac{\pi}{4}$$

$$= \boxed{\sqrt{2}}$$

V. Given that $\frac{dy}{d\theta} = -\sqrt{2}$, find the slope of the tangent line to the curve when $\theta = \frac{\pi}{4}$.

$$m_{tan} = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \quad \text{so} \quad \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \frac{-\sqrt{2}}{\sqrt{2}}$$

$$\boxed{m_{tan} = -1}$$

VI. Find the Cartesian description of the tangent line to the curve at $\theta = \frac{\pi}{4}$.

Express your final answer in the form $y = mx + b$.

$$y - y(\tfrac{\pi}{4}) = m_{tan}\left[x - x(\tfrac{\pi}{4})\right]$$

$$y - (-\sqrt{2}) = -1\left[x - (-\sqrt{2})\right]$$

$$y + \sqrt{2} = -x - \sqrt{2}$$

$$\boxed{y = -x - 2\sqrt{2}}$$

5. [25 pts] A curve C in space is given described by the vector-valued function:

$$\vec{r}(t) = \langle t^2 - 1, 2t^2 + 2, 2t \rangle .$$

I. Find $\vec{r}'(t)$.

$$\boxed{\vec{r}\,'(t) = \langle 2t, 4t, 2 \rangle}$$

II. Find a vector that is orthogonal to both $\vec{r}(0)$ and $\vec{r}'(0)$. Simplify your answer!

$\vec{r}(0) = \langle -1, 2, 0 \rangle$

$\vec{r}\,'(0) = \langle 0, 0, 2 \rangle.$

Such a vector is $\vec{r}(0) \times \vec{r}\,'(0) = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$= \boxed{4\hat{i} + 2\hat{j}}$ or $\boxed{\langle 4, 2, 0 \rangle}$

III. Find all points (x, y, z) on the curve at which $\vec{r}(t)$ is orthogonal to $\vec{r}'(t)$.

We need to find t so $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$

$\langle t^2-1, 2t^2+2, 2t \rangle \cdot \langle 2t, 4t, 2 \rangle = 0$

$(t^2-1)(2t) + (2t^2+2)(4t) + (2t)(2) = 0$

$2t^3 - 2t + 8t^3 + 8t + 4t = 0$

$10t^3 + 10t = 0$

$10t(t^2+1) = 0 \rightarrow \underline{t=0}.$

The point on the curve found using $\vec{r}(0) = \langle -1, 2, 0 \rangle$. $\rightarrow$ $\boxed{(x,y,z) = (-1,2,0)}$

(Parts D and E refer to the same function $\vec{r}(t) = \langle t^2 - 1, 2t^2 + 2, 2t \rangle$ on the last page!)

IV. Find the equation of the plane that passes through $(1, -2, 4)$ and whose normal vector is parallel to $\vec{r}(0)$. Express your final answer in the form $ax + by + cz = d$.

$\vec{n} = \vec{r}(0) = \langle -1, 2, 0 \rangle.$

The equation of a plane with normal vector $\vec{n} = \langle a, b, c \rangle$ passing through (x_0, y_0, z_0) is given by:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$-1(x-1) + 2(y - (-2)) + 0(z-4) = 0$$

$$-x + 1 + 2y + 4 + 0 = 0$$

$$\boxed{-x + 2y = -5}$$

V. Find an equation of the tangent line to the curve at $t = 0$.

The equation of a line parallel to $\vec{v}$ passing through P_0 is:

$\vec{R}(t) = \vec{v}t + \vec{P}_0.$

A vector parallel to the tangent line at $t = 0$ is:

$\vec{v} = \vec{r}\,'(0) = \langle 0, 0, 2 \rangle.$

The tangent line passes through $\vec{r}(0) = \langle -1, 2, 0 \rangle.$

So the tangent line is described by

$$\boxed{\vec{R}(t) = \langle 0, 0, 2 \rangle t + \langle -1, 2, 0 \rangle}$$

6. [10 pts] The position for a particle is described by the vector-valued function:

$$\vec{r}\,(t) = \left\langle \frac{t}{2}, \cos(at), \sin(at) \right\rangle, \quad t \geq 0.$$

Here, a is an unspecified constant.

I. Find the speed of the particle at $t = 0$ in terms of a.

$\vec{v}(t) = \vec{r}\,'(t) = \langle \frac{1}{2}, \ -a\sin at, a\cos at \rangle.$

Speed $= |\vec{v}(t)| = \sqrt{(\frac{1}{2})^2 + (-a\sin at)^2 + (a\cos at)^2}$

$= \sqrt{\frac{1}{4} + a^2\sin^2 at + a^2\cos^2 at}$

$= \sqrt{\frac{1}{4} + a^2}$ ← holds for all t!

so, $\boxed{|\vec{v}(0)| = \sqrt{\frac{1}{4} + a^2}}$

II. Find a value of a so that the curve is parameterized by arclength. Clearly *explain* why the curve is parameterized by arclength for your choice of a!

The curve is parameterized by arclength if and only if $|\vec{r}\,'(t)| = 1$ for all values of t!

Since $|\vec{r}\,'(t)| = \sqrt{\frac{1}{4} + a^2}$, the curve is parameterized by arclength if and only if

$$\sqrt{\frac{1}{4} + a^2} = 1$$

$$\frac{1}{4} + a^2 = 1$$

$$a^2 = \frac{3}{4}$$

$$\boxed{a = \pm\sqrt{\frac{3}{4}}}$$