Ross Program 2015 Application Problems

This document is part of the application to the *Ross Mathematics Program*, and is posted at http://u.osu.edu/rossmath. This challenging six-week residential program for high school students will run from June 14 to July 24, 2015.

The deadline for applications is May 1, 2015, but spaces fill as applications arrive. For adequate consideration of your application, it is best to submit it well before the end of April.

Each applicant should work independently on the problems below. We are interested in seeing how you approach unfamiliar math problems, not whether you can find answers by searching through web sites or books, or by asking experts.

Please submit your own work on all of these problems.

For each problem, explore the situation (with calculations, tables, pictures, etc.), observe patterns, make some guesses, test the truth of those conjectures, and describe the progress you have made. Where were you led by your experimenting?

Include your thoughts even though you may not have completely solved the problem. If you've seen one of the problems before (e.g. in a class or online), please include a reference along with your solution.

Please convert your problem solutions into a PDF file and upload them by following the instructions at `http://math.osu.edu/2015-ross-solutions`. That PDF file can be created by scanning your problem solutions from a handwritten paper copy. (It's best to use dark pencil or pen using only one side of the paper.) Alternatively, you could type up the solutions using a word processor or with LaTeX, and then convert the output to PDF format.

Note: each Ross Program course concentrates deeply on one subject, unlike the problems here. This Problem Set is an attempt to assess your general mathematical background and interests.

Problem 1

A set of numbers has the *triple-sum property* (or TSP) if there exist three numbers in the set whose sum is also in the set. Repetitions are allowed. For example, the set $U = \{2, 3, 7\}$ has TSP since $2 + 2 + 3 = 7$, while $V = \{2, 3, 10\}$ fails to have TSP.

(a) Suppose the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ is separated into two parts, forming two subsets A and B. Prove that either A or B must have the triple-sum property.

> To begin, suppose that statement false, so there are sets A and B as above, each without TSP. If 1 lies in A then $3 = 1+1+1$ must be in B. Continue until you find an impossibility.

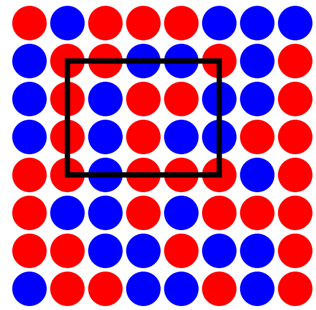
(b) Is a similar result true when the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is separated into two parts?

Problem 2

Suppose 64 dots are arranged in a square 8-by-8 array, and each dot is colored red or blue.

(a) Prove that this array must contain a "monochromatic" rectangle. That is, no matter how the red and blue colors are assigned, there must be either a set of four red dots that form a rectangle or else a set of four blue dots that form a rectangle.

Consider just the four corner points here, not the colors of dots inside that rectangle. Restrict attention to those rectangles with horizontal and vertical sides.



A monochromatic rectangle.

(b) Consider the following property of m and n:

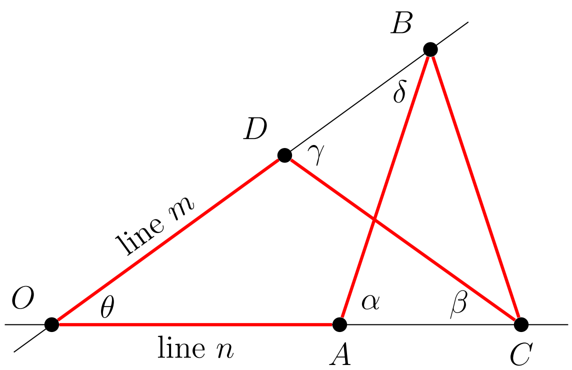
Every m-by-n array of red and blue dots has a monochromatic rectangle.

Part (a) shows that this property is true for size 8-by-8. Find an example of a 4-by-5 array of red and blue dots that possesses no monochromatic rectangle. This shows that the property above is false for 4-by-5 arrays. For which m-by-n arrays is the property true? For example, is the property true for 4-by-6 arrays? For 5-by 5-arrays?

(c) If each dot in a 100-by-100 array is colored one of three colors (red, white, or blue), must that array contain a monochromatic rectangle?

Problem 3

Two lines m and n meet at point O, making an angle $\theta \le 90°$ there. (More precisely, θ is the measure of that angle in degrees).



Rusty the Robot walks in the plane with steps that alternate between those lines m and n, starting from the intersection point O. Rusty is old and has stiff legs so all his steps have the same length (but they may be in any direction). Since Rusty is not allowed to return to a spot he just occupied, he might become stuck with no legal step to make. We can see Rusty's path because he leaves a trail of red rust behind him as he walks.

If $\theta = 90°$, then Rusty is stuck after just one step.

If $\theta = 60°$, then Rusty returns to O after 3 steps.

(a) For the angle θ pictured above, Rusty steps from O to A to B to C to D to O, returning to O in 5 steps. What is θ in this case?

> Idea: Segments $\overline{OA}$ and $\overline{AB}$ and $\overline{BC}$ and $\overline{CD}$ and $\overline{DO}$ are the same length. As a consequence, angles θ and β are equal because ΔODC is isosceles. This implies that $\gamma = 2\theta$. Continue this argment to show that $\theta = 36°$.

(b) For which angle θ will Rusty return to O after 7 steps? What's the general rule? If Rusty returns to O after k steps, find (with proof!) the angle θ.

If you enjoy this problem, you might like to explore the following: for the $36°$ situation in the picture above, suppose Rusty starts at some point P close to O on line m. What will his path be? Will Rusty return to his starting point P after some number of steps?

Problem 4

A polynomial $f(x)$ has the *factor-square property* (or FSP) if $f(x)$ is a factor of $f(x^2)$. For instance, $g(x) = x - 1$ and $h(x) = x$ have FSP, but $k(x) = x + 2$ does not.

Reason: $x - 1$ is a factor of $x^2 - 1$, and x is a factor of x^2, but $x + 2$ is not a factor of $x^2 + 2$.

Multiplying by a nonzero constant "preserves" FSP, so we restrict attention to polynomials that are *monic* (i.e., have 1 as highest-degree coefficient).
Check that x^2, $x^2 - x$, $x^2 - 2x + 1$, and $x^2 + x + 1$ all have FSP.

What is the pattern to these FSP polynomials? To help you make progress on this general question, investigate the following questions and justify your answers.

(a) Are x and $x - 1$ the only monic polynomials of degree 1 with FSP?

(b) Which monic polynomials of degree 2 have FSP?

(c) Examples of larger degree can be built as products of earlier examples. For instance, the following polynomials of degree 3 all have FSP:

$$x^3 + x^2 + x = x(x^2 + x + 2),$$
$$x^3 - 1 = (x - 1)(x^2 + x + 1),$$
$$x^3 - x^2 = x^2(x - 1).$$

Are there monic FSP polynomials of degree 3 that are not built from FSP polynomials of degree 1 or 2? Similarly, are there examples of degree 4 monic polynomials with FSP that do not arise as products of FSP polynomials of smaller degrees?

(d) The examples written above all had integer coefficients. Do answers change if we allow polynomials whose coefficients are allowed to be any real numbers? Or if we allow polynomials whose coefficients are complex numbers?

Problem 5

A point $A = (m, n)$ is a "lattice point" when both m and n are integers. Let's call a point $P = (x, y)$ "generic" if all the distances from P to lattice points are different. With some algebraic work, I checked that the point $S = (\sqrt{2}, \sqrt{3})$ is generic. However, the point $T = (0, \pi)$ is not generic because it is equally distant from the lattice points $(1, 0)$ and $(-1, 0)$.

Question. Is there some generic point with rational coordinates?

That is, if $Q = (r, s)$ for rational numbers r and s, must there exist two lattice points equidistant from Q? As a first step, check that $R = \left(\frac{3}{4}, \frac{2}{5}\right)$ is not generic. (Find lattice points A, B equidistant from R.)

Can you use those ideas to answer the general question?

Problem 6

What numbers can be expressed as an alternating-sum of an increasing sequence of powers of 2? To form such a sum, choose a subset of the sequence $1, 2, 4, 8, 16, 32, 64, \ldots$ (these are the powers of 2). List the numbers in that subset in increasing order (no repetitions allowed), and combine them with alternating plus and minus signs. For example,

$$\begin{array}{llll} 1 = -1 + 2; & 2 = -2 + 4; & 3 = 1 - 2 + 4; & \\ 4 = -4 + 8; & 5 = 1 - 4 + 8; & 6 = -2 + 8; & \text{etc.} \end{array}$$

Note: the expression $5 = -1 - 2 + 8$ is invalid because the signs are not alternating.

(a) Is every positive integer expressible in this fashion? If so, give a convincing proof.

(b) A number might have more than one expression of this type. For instance

$$3 = 1 - 2 + 4 \quad \text{and} \quad 3 = -1 + 4.$$

Given a number n, how many different ways are there to write n in this way? Explain why your answer is correct.

Problem 7

Given a "seed" number c, define a sequence $\{b_n\}$ as follows:

$$b_1 = c, \quad b_2 = c - \frac{1}{c}, \quad b_3 = c - \frac{1}{c - \frac{1}{c}}, \qquad \text{etc.}$$

That sequence $\{b_n\}$ is defined recursively by setting $b_1 = c$ and

$$b_{n+1} = c - \frac{1}{b_n} \quad \text{for every index } n \geq 1.$$

I used my calculator to find the first few terms of those sequences $\{b_n\}$ for the two seed values $c = 3$ and $c = 3/2$.

$c = 3$	$c = 3/2$
$b_1 = 3 = 3.000$	$b_1 = 3/2 = 1.500$
$b_2 = 8/3 \approx 2.667$	$b_2 = 5/6 \approx 0.833$
$b_3 = 21/8 = 2.625$	$b_3 = 3/10 = 0.300$
$b_4 = 55/21 \approx 2.619$	$b_4 = -11/6 \approx -1.833$
$b_5 = 144/55 \approx 2.618$	$b_5 = 45/22 \approx 2.045$

When $c = 3$, the sequence $\{b_n\}$ appears to have all positive terms, with values decreasing steadily. That sequence seems to stabilize, ending with $b_n \approx 2.618033989$ for values $n > 12$ (using a calculator with nine decimal place accuracy).

When $c = 3/2 = 1.5$, the sequence $\{b_n\}$ is not always positive, does not steadily decrease, and does not seem to stabilize.

Further numerical experiments seem to say that the first behavior happens when $c > 2$, and the second happens when $c < 2$.

Question. If $c \geq 2$ must the sequence $\{b_n\}$ be decreasing, and always positive?
If $0 < c < 2$ must the sequence $\{b_n\}$ contain some negative values, or else degenerate because $b_k = 0$ for some k? Justify your answers.

Note: numerical calculations alone cannot provide a full explanation (proof) of your observations.

Problem 8

Which of the problems here did you enjoy the most? Why?

We hope you enjoyed working on these problems! Information about this summer mathematics program is available on the web at http://u.osu.edu/rossmath/. Your questions and comments can be emailed to ross@math.ohio-state.edu.