## DISCLAIMER

### General Information:

This midterm is a *sample* midterm. This means:

- The sample midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to the actual midterm. Note that this may be a departure from the format used on exams in previous semesters!
- The sample midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

### How to take the sample exam:

The sample midterm should be treated like the actual exam. This means:

- "Practice like you play." Schedule 55 uninterrupted minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.
- The problems on this exam are mostly based on the Worksheets posted on the Math 1172 website and your previous quizzes.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  Thus, there will be a problem that will require you to apply the material in an unfamiliar setting. To aid in preparation, there is such a problem on this sample exam.

### How to use the solutions:

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

Math 1172 Name: ______________________

Sample Midterm 1 OSU Username (name.nn): ______________________

Spring 2016 Lecturer: ______________________

Recitation Instructor ______________________

Form B Recitation Time: ______________________

## Instructions

- You have **55 minutes** to complete this exam. It consists of 6 problems on 10 pages including this cover sheet. Page 10 may be used extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work!** Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be xeroxed prior to being returned.

| Problem | Point Value | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 10 | |
| 3 | 20 | |
| 4 | 15 | |
| 5 | 20 | |
| 6 | 15 | |
| Total | 100 | |

1. **Multiple Choice** [20 pts]

   Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

   I. The magnitude of the force experienced by a particle moving along the $x$-axis is given by $F(x) = kx^2$, where $F$ is measured in Newtons (N) and $x$ is measured in meters (m). Find the total work done by this variable force in moving a particle from $x = 1$ to $x = 2$ if $k = 6$ N/m.

   $\boxed{\text{A. 14 J}}$    B. 9 J    C. 8 J    D. None of the above

   E. This is impossible to determine with the given information!

   $$W = \int_1^2 F(x)\,dx = \int_1^2 6x^2\,dx$$
   $$= 2x^3\Big|_1^2$$
   $$= 2(2)^3 - 2(1)^3 = \underline{14}$$

   II. A thin wire is represented by a line segment on the interval $x = 0$ to $x = \frac{\pi}{2}$. Find the mass of the wire if its density is given by $\rho(x) = \dfrac{8}{x^2+4}$.

   A. $m = 8\ln\left(\frac{\pi^2}{16} + 4\right) - 8\ln 4$    B. $m = 8\arctan\left(\frac{\pi}{2}\right) - 8\arctan 0$

   C. $m = 8$    D. $m = 4$

   E. The mass is infinite.    $\boxed{\text{F. None of the above}}$

   $$m = \int_0^{\pi/2} \frac{8}{x^2+4}\,dx = 8 \cdot \frac{1}{2}\arctan\frac{x}{2}\Big|_0^{\pi/2}$$
   $$= 4\arctan\frac{\pi}{4} - 4\cancel{\arctan 0}^{\,0}$$
   $$= \underline{4\arctan\frac{\pi}{4}} \leftarrow \quad \arctan\frac{\pi}{4} \neq 1!$$
   $$\left(\tan\frac{\pi}{4} = 1\right).$$

   $$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C}$$

   III. Find a function $f(x)$ that satisfies $\int f(x)\,dx = x^2\sin x + C$

   A. $f(x) = 2x\cos x$    $\boxed{\text{B. } f(x) = 2x\sin x + x^2\cos x}$

   C. $f(x) = 2x\sin x - x^2\cos x$    D. No such function exists

   $$f(x) = \frac{d}{dx}\left[x^2\sin x\right] \leftarrow \text{Use product rule!}$$
   $$= 2x\sin x + x^2\cos x$$

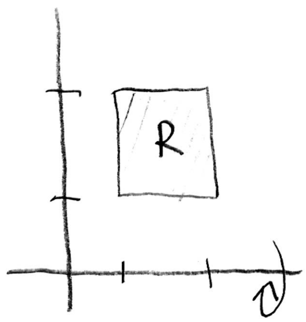
IV. Let $R$ be the region bounded by the lines $x = 1$, $x = 2$, $y = 1$, and $y = 2$. Let $V_1$ be the volume of the region obtained by revolving $R$ about the $x$-axis and $V_2$ be the volume of the region obtained by revolving $R$ about the $y$-axis. Which of the following best describes the relationship between $V_1$ and $V_2$?

A. $V_1 < V_2$

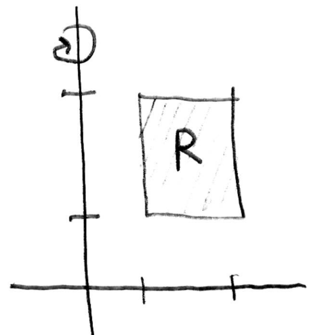
$\boxed{\text{B. } V_1 = V_2}$

C. $V_1 > V_2$

D. Not enough information is provided to determine this.




Generates $V_1$




Generates $V_2$

Symmetry ⇒ these volumes are equal!

V. Two identical cylindrical tanks, Tank 1 and Tank 2, are filled to heights $h_1$ and $h_2$, respectively. Tank 1 is filled with Liquid X ($\rho = \rho_1$) while Tank 2 is filled with Liquid Y ($\rho = \rho_2$).

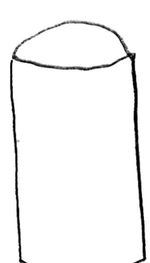
Suppose that the work required to pump the liquid out of each tank is equal. If $\rho_1 > \rho_2$, which of the following best describes the relationship between $h_1$ and $h_2$?

$\boxed{\text{A. } h_1 < h_2}$

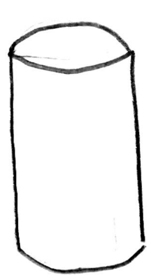
B. $h_1 = h_2$

C. $h_1 > h_2$

D. Not enough information is provided to determine this.



Tank 1



Tank 2

Liquid in Tank 1 is heavier so if the same amount of work is done to pump liquid out of both tanks, Tank 2 must have more liquid!

2. **Short Answer**

I. [5 pts] The length $L$ of the segment of the curve $y = x^2$ from $x = 1$ to $x = 2$ can be expressed as either an integral with respect to $x$ or as an integral with respect to $y$.

With respect to $x$, the length of the segment is given by:

$$L = \int_1^2 \sqrt{1 + 4x^2}\, dx. \tag{1}$$

With respect to $y$, the length of the segment is given by:

$$L = \int_1^4 \sqrt{1 + \frac{1}{4y}}\, dy. \tag{2}$$

By making the substitution $y = x^2$ into Eqn (2), show that the integral in Eqn. (2) transforms into the integral given in Eqn (1).

$y = x^2 \quad \rightarrow \quad y = 1 \rightarrow x = 1$

$dy = 2x\, dx. \quad y = 4 \rightarrow x = 2.$

So $L = \int_{y=1}^{y=4} \sqrt{1 + \frac{1}{4y}}\, dy = \int_{x=1}^{x=2} \sqrt{1 + \frac{1}{4x^2}} \cdot 2x\, dx$

$= \int_{x=1}^{x=2} \sqrt{1 + \frac{1}{4x^2}} \sqrt{4x^2}\, dx$

$= \int_{x=1}^{x=2} \sqrt{4x^2 + 1}\, dx$ ✓

II. [5 pts] A student claims that for $x > 1$:

$$\int \frac{1}{x^2 - 1}\, dx = \ln(x^2 - 1) + C.$$

Briefly explain why the student is either correct or incorrect. To receive full credit, you must justify your answer!

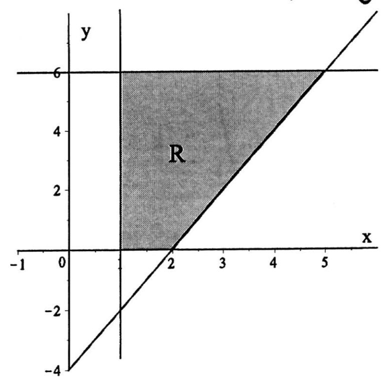
The student is incorrect! If the student were correct

$$\frac{1}{x^2-1} = \frac{d}{dx} \ln(x^2+1)$$

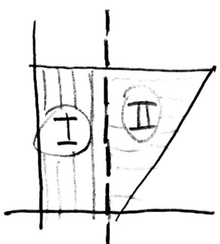
but $\frac{d}{dx} \ln(x^2-1) = \frac{1}{x^2-1} \cdot 2x$ !

3. [20 pts] The region $R$ is bounded by the curves $y = 2x - 4$, $x = 1$, $y = 0$ and $y = 6$.

$x = 2 + \frac{1}{2}y$



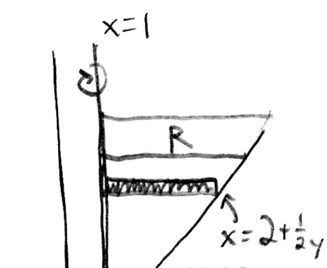

I. Set up, but do not evaluate, an integral or a sum of integrals **with respect to $x$** that would give the area of $R$.




$$\boxed{A = \int_1^2 6\,dx + \int_2^5 [6 - (2x-4)]\,dx}$$

II. Suppose now that a tank is formed by revolving the region $R$ about the line $x = 1$ and $x$ and $y$ are measured in meters.

A. Set up, but do not evaluate, an integral or a sum of integrals that would give the volume of the tank.




- We want to integrate with respect to y!
  - → Use horizontal strips.
- Horizontal strips are perpendicular to the axis of rotation
  - → Use Washer Method!

$R = 2 + \frac{1}{2}y - 1 = 1 + \frac{1}{2}y$

$$V = \int_{y=0}^{y=6} \pi(R^2 - \overset{0}{\cancel{r^2}})\,dy = \boxed{\int_0^6 \pi\left(1 + \tfrac{1}{2}y\right)^2 dy}$$

B. Suppose the tank is now filled to a height $y = 3$ with water ($\rho = 1000$ kg/m$^3$). Set up, but do not evaluate, an integral or a sum of integrals that would give thework required to pump the liquid out of the tank. You may assume $g = 9.8$ m/s$^2$.

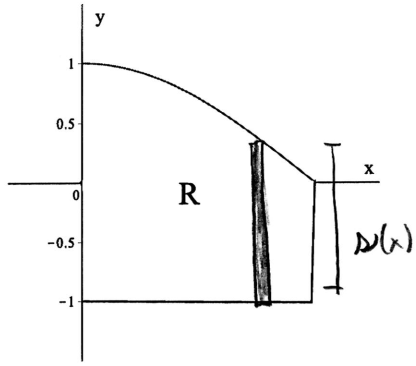
$$W = \int_0^3 \rho g\, A(y)\, D(y)\, dy$$

← $A(y)$ is the area of a circle of radius $R = 1 + \frac{1}{2}y$.

→ $A(y) = \pi\left(1 + \frac{1}{2}y\right)^2$

$$= \boxed{\int_0^3 1000(9.8)\pi\left(1 + \tfrac{1}{2}y\right)^2 (6-y)\,dy}$$

4. [15 pts] The base of a solid is the region in the $xy$-plane bounded by the curves $y = \cos\left(\frac{x}{2}\right)$ , $y = -1$ , $x = 0$ , and $x = \pi$ is shown below.




If cross-sections through the solid that are perpendicular to the $x$-axis are squares, find the volume of the solid.

$$V = \int_0^{\pi} [A(x)]^2\,dx$$

Here $A(x) = [s(x)]^2 = \left[\cos\frac{x}{2} - (-1)\right]^2$

$$= \cos^2\frac{x}{2} + 2\cos\frac{x}{2} + 1$$

So: $V = \int_0^{\pi} \left(\cos^2\frac{x}{2} + 2\cos\frac{x}{2} + 1\right) dx$

$\downarrow \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$$= \int_0^{\pi} \left[\left(\frac{1}{2} + \frac{1}{2}\cos x\right) + 2\cos\frac{x}{2} + 1\right] dx$$

$$= \int_0^{\pi} \left[\frac{3}{2} + \frac{1}{2}\cos x + 2\cos\frac{x}{2}\right] dx$$

$$= \left[\frac{3}{2}x + \frac{1}{2}\sin x + 4\sin\frac{x}{2}\right|_0^{\pi}$$

$$= \left[\frac{3}{2}\pi + \frac{1}{2}\cancel{\sin\pi}^{\,0} + 4\sin\frac{\pi}{2}\right] - [0]$$

$$= \left(\frac{3}{2}\pi + 4\right) - 0$$

$$= \boxed{\frac{3}{2}\pi + 4}$$

5. [20 pts] Evaluate the following antiderivatives.

I. $\int_0^1 8xe^{2x}\,dx$

$u = 8x \qquad dv = e^{2x}\,dx$

$du = 8\,dx \qquad v = \frac{1}{2}e^{2x}$

$$\int_0^1 8xe^{2x}\,dx = 4xe^{2x}\Big|_0^1 - \int_0^1 4e^{2x}\,dx$$

$$= \left[4(1)e^{2(1)} - \overset{0}{\cancel{4(0)e^{2(0)}}}\right] - 2e^{2x}\Big|_0^1$$

$$= 4e^2 - \left[2e^2 - 2e^0\right]$$

$$= \boxed{2e^2 + 2}$$

II. $\int \frac{4}{(4+x^2)^{3/2}}\,dx$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

So: $$\int \frac{4}{(4+x^2)^{3/2}}\,dx = \int \frac{4}{(4+4\tan^2\theta)^{3/2}} \cdot 2\sec^2\theta\,d\theta$$

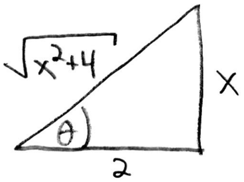
$$= \int \frac{4}{8\sec^3\theta} \cdot 2\sec^2\theta\,d\theta$$

$$= \int \frac{1}{\sec\theta}\,d\theta$$

$$= \int \cos\theta\,d\theta$$

$$= \sin\theta + C$$

$$= \boxed{\frac{x}{\sqrt{x^2+4}} + C}$$




$x = 2\tan\theta$

$\tan\theta = \frac{x}{2}$

6. [15 pts] (The Leaky Bucket Problem)

*Jack and Jill went up a hill*
*To fetch a pail of water...*

In fact, their pail of water carries exactly 10 kg (about 3 gallons) of water when filled. They lower their pail into a well that is 10 m deep and it becomes filled with water. Together, they pull the pail of water up at a constant rate of .5 m/sec.

Unfortunately for Jack and Jill, the pail that they are using has a small hole in the bottom that causes water to leak out at a constant rate of .1 kg /s. Thus, the formula:

$$\text{(Work)} = \text{(force)} \cdot \text{(displacement)}$$

cannot be applied here since the mass of the bucket is changing (and hence the force due to gravity is variable).

Calculate the amount of work Jack and Jill do to raise the pail from the bottom of the well to the top of the well by following the steps below.

I. Let $y(t)$ indicate the position of the bucket at time $t$, and set $y(0) = 0$.

A. Write down the function $y(t)$.

$\boxed{y(t) = .5t}$

B. Write an expression for the total mass of water that **remains in the bucket** after a time $t$ in terms of $y$.

mass = (initial mass) − (mass of water out)

$m(t) = 10 - .1t.$

Since $y(t) = .5t$, $t = 2y$, so:

$m(y) = 10 - .1(2y)$

$\boxed{m(y) = 10 - .2y}$

(*Problem 6 continued* )

II. During a small time interval from $t$ to $t + \Delta t$, the work required to move the bucket over the corresponding distance $\Delta y$ can be approximated by assuming that the mass of water in that bucket, and hence the force, is constant.

A. Write down the amount of work, $\Delta W$, required to move the bucket over the small interval $\Delta y$. You may assume that the force is given by $F(y) = m(y)g$, where $g = 9.8$ m/s and $m(y)$ is the mass you found in I, part B.

$$F = (10 - .2y)g$$

Since the mass is appx constant over $\Delta y$,

$$\Delta W = F \, \Delta y$$

$$\boxed{\Delta W = (10 - .2y)g \, \Delta y}$$

B. Write down an integral that gives the total work $W$ necessary to raise the leaky bucket to the top of the well.

$$\boxed{W = \int_{y=0}^{y=10} (10 - .2y)g \, dy}$$