------------------ **DISCLAIMER** ------------------

General Information:

This midterm is a *sample* midterm. This means:

- The sample midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to the actual midterm. Note that this may be a departure from the format used on exams in previous semesters!
- The sample midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take the sample exam:

The sample midterm should be treated like the actual exam. This means:

- "Practice like you play." Schedule 55 uninterrupted minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.
- The problems on this exam are mostly based on the Worksheets posted on the Math 1152 website and your previous quizzes.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and take-home quizzes with this intent, and there will be a problem on your midterm that will require you to apply the material in an unfamiliar setting. To aid in preparation, there is such a problem on this sample exam.

How to use the solutions:

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

Math 1152

Sample Midterm 2

Autumn 2016

Form A

Name: Solutions

OSU Username (name.nn):

Lecturer:

Recitation Instructor:

Recitation Time:

Instructions

- You have **55 minutes** to complete this exam. It consists of 6 problems on 10 pages including this cover sheet. Page 10 may be used for extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	12	
2	18	
3	30	
4	40	
Total	100	

1. **Multiple Choice** [12 pts]

Circle the response that best answers each question. Each question is worth 3 points. There is no penalty for guessing and no partial credit.

I. If $\sum_{k=1}^{\infty} a_k = 5$ and $\sum_{k=1}^{\infty}(a_k + b_k) = 3$, what is $\lim_{n\to\infty} b_n$?

$\boxed{\text{A. } 0}$ B. -2 C. Does not exist

D. This cannot be determined unless we have a formula for b_n.

E. None of the above.

By properties of convergent series:

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}(a_k + b_k) - \sum_{k=1}^{\infty} a_k = 3-5 = -2.$$

So the series $\sum_{k=1}^{\infty} b_k$ converges to -2. Since $\sum b_k$ converges, the limit of the sequence b_n must be 0! i.e. $\lim_{n\to\infty} b_n = 0$

II. Which of the following series is equivalent to $\sum_{k=1}^{\infty} \frac{2}{k^2+1}$?

A. $\sum_{k=0}^{\infty} \frac{2}{k^2+2k+2}$

B. $\sum_{k=0}^{\infty} \frac{2}{(k-1)^2+1}$

C. $\sum_{k=2}^{\infty} \frac{2}{(k+1)+1}$

D. None of the above

Let $m = k-1$ so $k=1 \leftrightarrow m=0$. Then, $k = m+1$ and:

$$\sum_{k=1}^{\infty} \frac{2}{k^2+1} = \sum_{m=0}^{\infty} \frac{2}{(m+1)^2+1} = \sum_{m=0}^{\infty} \frac{2}{m^2+2m+2}$$

$$= \sum_{k=0}^{\infty} \frac{2}{k^2+2k+2}$$

The name you give to the index doesn't matter!

III. Which of the following is **necessary** for the Ratio Test to be applied to $\sum_{k=0}^{\infty} a_k$?

A. $a_k \geq 0$ for all k eventually.

B. a_k is decreasing for all k eventually.

C. $\lim_{k\to\infty} a_k = 0$.

D. $\sum_{k=0}^{\infty} a_k$ is an alternating series.

E. More than one of these

F. None of these

IV. Consider the series $\sum_{k=0}^{\infty} a_k$ and suppose $a_k \geq 0$ for all $k \geq 0$. Of the following options:

i. Converge absolutely

ii. Converge Conditionally

iii. Diverge

which option below most correctly describes the possibilities for this series?

A. i. only

B. ii. only

C. iii. only

D. Both i. and iii.

E. Both ii. and iii.

F. i, ii, and iii.

- Note if $a_k \geq 0$ for all k, $|a_k| = a_k$ so if $\sum_{k=0}^{\infty} a_k$ converges $\sum |a_k|$ must converge! So, if the series converges, it must do so absolutely!
(The series cannot converge conditionally; recall $\sum a_k$ converges conditionally if $\sum a_k$ converges but $\sum |a_k|$ diverges)
- Of course, $\sum a_k$ could still diverge as well

2. **Short Answer** [18 pts]

Determine whether the following statements are **True** or **False** and briefly explain your response.

I. [6 pts] *The function* $\frac{1}{\sqrt{x}}$ *is unbounded at* $x = 0$, *so the improper integral* $\int_0^2 \frac{1}{\sqrt{x}}\,dx$ *diverges.*

False; $\int_0^2 \frac{1}{\sqrt{x}}$ diverges if and only if $\lim_{a\to 0^+} \int_a^2 \frac{1}{\sqrt{x}}\,dx$ DNE. But,

$$\lim_{a\to 0^+} \int_a^2 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^2 x^{-1/2}\,dx = \lim_{a\to 0^+} \left[2x^{1/2}\right]_a^2 = \lim_{a\to 0^+} \left[2\sqrt{2}-2\sqrt{a}\right]$$
$$= 2\sqrt{2}.$$

Hence, the improper integral converges to $2\sqrt{2}$.

II. Suppose $\sum_{k=1}^{\infty} a_k = 3$ and $b_n = 2$ for all $n \geq 1$. Then:

A. [6 pts] $\sum_{k=1}^{\infty}(a_k - 3) = 0$.

False; $\sum_{k=1}^{\infty} a_k$ converges so $\lim_{k\to\infty} a_k = 0$ Hence,

$\lim_{k\to\infty} (a_k - 3) = -3 \neq 0$ so $\sum_{k=1}^{\infty} (a_k - 3)$ diverges by the divergence test!

B. [6 pts] $\sum_{k=1}^{\infty}(b_k - 2) = 0$.

True; let $s_n = \sum_{k=1}^{n} (b_k - 2)$. Since $b_k = 2$ for all k,

$b_k - 2 = 0$ for all k and $s_n = \sum_{k=1}^{n} 0 = 0$ for all n.

Hence, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} 0 = 0$ so $\sum_{k=1}^{\infty} (b_k - 2) = 0$.

3. Evaluate the following antiderivatives or find the definite integrals.

I. [10 pts] $\int \tan^2\left(\frac{\theta}{3}\right)\sec^6\left(\frac{\theta}{3}\right)\,d\theta$.

- If $u = \sec\frac{\theta}{3}$, need $\tan\frac{\theta}{3}\sec\frac{\theta}{3}$ for du, which leaves an odd # of $\tan\frac{\theta}{3}$ ✗
- If $u = \tan\frac{\theta}{3}$, need $\sec^2\frac{\theta}{3}$ for du, which leaves an even # of $\sec\frac{\theta}{3}$ ✓

$du = \frac{1}{3}\sec^2\frac{\theta}{3}\,d\theta$.

$\frac{3\,du}{\sec^2\frac{\theta}{3}} = d\theta$

Thus, $\int \tan^2\frac{\theta}{3}\sec^6\frac{\theta}{3}\,d\theta = \int u^2\cdot\sec^6\frac{\theta}{3}\,\frac{3\,du}{\sec^2\frac{\theta}{3}} = \int 3u^2\sec^4\frac{\theta}{3}\,du$

$\sec^4\frac{\theta}{3} = \left(1+\tan^2\frac{\theta}{3}\right)^2$

Using $\sec^2\frac{\theta}{3} = 1+\tan^2\frac{\theta}{3}$:

$$\int \tan^2\frac{\theta}{3}\sec^6\frac{\theta}{3}\,d\theta = \int 3u^2(1+u^2)^2\,du$$
$$= \int 3u^2(1+2u^2+u^4)\,du$$
$$= \int (3u^2+6u^4+3u^6)\,du$$
$$= u^3+\frac{6}{5}u^5+\frac{3}{7}u^7+C$$
$$= \boxed{\tan^3\frac{\theta}{3}+\frac{6}{5}\tan^5\frac{\theta}{3}+\frac{3}{7}\tan^7\frac{\theta}{3}+C}$$

II. [10 pts] $\int_0^\infty xe^{-2x}\,dx$.

$= \lim_{b\to\infty}\int_0^b xe^{-2x}\,dx$

$u = x \quad dv = e^{-2x}$

$du = dx \quad v = -\frac{1}{2}e^{-2x}$

$= \lim_{b\to\infty}\left[-\frac{1}{2}xe^{-2x}\Big|_0^b - \int_0^b -\frac{1}{2}e^{-2x}\,dx\right]$.

$= \lim_{b\to\infty}\left[-\frac{1}{2}be^{-2b} - 0 - \frac{1}{4}\left[e^{-2x}\Big|_0^b\right]\right]$

By L'Hopital's rule

$= \lim_{b\to\infty}\left[-\frac{b}{2e^{2b}} - \frac{1}{4}e^{-2b}+\frac{1}{4}\right]$

$\lim_{b\to\infty}\frac{b}{e^{2b}} \overset{L'H}{=} \lim_{b\to\infty}\frac{1}{2e^{2b}} = 0$.

$= 0 - 0 + \frac{1}{4}$.

$\boxed{\text{The integral converges to } \frac{1}{4}}$

III. [10 pts] $\int \frac{4x^3+3x^2+1}{x^4+x^2}\,dx.$ ← Can factor denom → try Partial Fractions!

$$\frac{4x^3+3x^2+1}{x^2(x^2+1)} = \frac{A}{x^2}+\frac{B}{x}+\frac{Cx+D}{x^2+1}$$

$$4x^3+3x^2+1 = A(x^2+1)+Bx(x^2+1)+x^2(Cx+D).$$

$\underline{x=0}$: $\underline{1 = A}$.

So $4x^3+3x^2+1 = 1(x^2+1)+Bx(x^2+1)+x^2(Cx+D)$

$$4x^3+3x^2+\cancel{1} = x^2+\cancel{1}+Bx^3+Bx+Cx^3+Dx^2$$

$$4x^3+2x^2 = (B+C)x^3+Dx^2+Bx.$$

So: $\begin{cases} 4=B+C \\ \underline{2=D} \\ \underline{0=B} \quad \Rightarrow \quad \underline{C=4} \end{cases}$

Thus, $\frac{4x^3+3x^2+1}{x^2(x^2+1)} = \frac{1}{x^2}+\frac{4x+2}{x^2+1}$

and $\int \frac{4x^3+3x^2+1}{x^4+x^2}\,dx = \int\left(\frac{1}{x^2}+\frac{4x+2}{x^2+1}\right)dx$

$$= \int\left(x^{-2}+\underbrace{\frac{4x}{x^2+1}}_{u=x^2+1}+\frac{2}{x^2+1}\right)dx$$

$$= -x^{-1}+2\ln|x^2+1|+2\arctan x+C.$$

$$= \boxed{-\frac{1}{x}+2\ln(x^2+1)+2\arctan x+C}.$$

4. [40 pts] Determine whether each series below converges or diverges. In order to get full credit, you must *fully* justify your work by:

- **Stating** any convergence test you use and why the test applies.
- **Explaining** the conclusions of the test!

  You may quote any results about p-series or geometric series, but you must *clearly* explain why the type of series you are considering converges or diverges!

I. $\displaystyle\sum_{k=3}^{\infty} \frac{2k^2}{k^5+1}$ ← This is a rational function, in k, so try limit comparison test!

- Since $\frac{2k^2}{k^5+1} \geq 0$ for all k, we can apply limit comparison test.
- Since $\lim_{k\to\infty} \left(\frac{2k^2}{k^5+1}\right) / \left(\frac{2}{k^3}\right) = \lim_{k\to\infty} \frac{\cancel{2}k^2}{k^5+1} \cdot \frac{k^3}{\cancel{2}} = \lim_{k\to\infty} \frac{k^5}{k^5+1} = 1$

  the limit comparison test ensures that $\sum \frac{2k^2}{k^5+1}$ and $\sum \frac{1}{k^3}$ will either both converge or both diverge.
- Since $\sum \frac{1}{k^3}$ is a p-series with $p > 1$, it converges
- Hence, $\sum \frac{2k^2}{k^5+1}$ converges.

II. $\displaystyle\sum_{k=1}^{\infty} (-1)^k \left[\frac{2+k^2}{k+2k^2}\right]^k$ ← This is alternating, but it would be annoying to check that $\left(\frac{2+k^2}{k+k^2}\right)^k$ is decreasing!

Check for absolute convergence; i.e. check if:

$$\sum_{k=1}^{\infty} \left| (-1)^k \left[\frac{2+k^2}{k+2k^2}\right]^k \right| = \sum_{k=1}^{\infty} \left[\frac{2+k^2}{k+2k^2}\right]^k$$

converges. Note, $\left[\frac{2+k^2}{k+2k^2}\right]^k > 0$ for all k, so we can now apply Root Test.

$$L = \lim_{k\to\infty} \sqrt[k]{\left[\frac{2+k^2}{k+2k^2}\right]^k} = \lim_{k\to\infty} \frac{2+k^2}{k+2k^2} = \frac{1}{2}.$$

Since $L < 1$, $\displaystyle\sum_{k=1}^{\infty} \left[\frac{2+k^2}{k+2k^2}\right]^k$ converges by the Root Test.

Hence, $\sum (-1)^k \left[\frac{2+k^2}{k+2k^2}\right]^k$ converges absolutely, so it converges.

III. $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$

Since $\frac{\ln k}{k^2} > 0$ for all k, we can use the comparison test.

Note $\lim_{k \to \infty} \frac{\ln k}{k^{1/2}} = 0$ so for all large enough k, $\frac{\ln k}{k^{1/2}} < 1$

$\ln k < k^{1/2}$

Hence, for all large k, $\frac{\ln k}{k^2} < \frac{k^{1/2}}{k^2} = \frac{1}{k^{3/2}}$

Since $\sum \frac{1}{k^{3/2}}$ is a p-series with $p > 1$, it converges.

Hence, $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ converges by the comparison test.

IV. $\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right)$.

Note: $\ln \frac{k}{k+1} = \ln k - \ln(k+1)$, so this is telescoping!

We need to find a formula for s_n so we can determine if $\lim_{n \to \infty} s_n$ exists.

Letting $a_k = \ln k - \ln(k+1)$ and $s_n = \sum_{k=1}^{n} a_k$:

$a_1 = \cancel{\ln 1}^{0} - \ln 2 \rightarrow s_1 = -\ln 2$

$a_2 = \ln 2 - \ln 3$ $\quad s_2 = -\ln 3$

$a_3 = \ln 3 - \ln 4$ $\quad s_3 = -\ln 4$

$a_4 = \ln 4 - \ln 5$ $\quad s_4 = -\ln 5$

$\vdots$ $\quad \vdots$

$a_n = \ln n - \ln(n+1)$ $\quad s_n = -\ln(n+1)$.

Hence, $\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left[-\ln(n+1)\right] = -\infty$

Since $\lim_{n \to \infty} s_n$ DNE, $\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right)$ diverges.