DISCLAIMER

General Information:

This midterm is a *sample* midterm. This means:

- The sample midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to the actual midterm. Note that this may be a departure from the format used on exams in previous semesters!
- The sample midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take the sample exam:

The sample midterm should be treated like the actual exam. This means:

- "Practice like you play." Schedule 55 uninterrupted minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.
- The problems on this exam are mostly based on the Worksheets posted on the Math 1172 website and your previous quizzes.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and take-home quizzes with this intent, and there will be a problem on your midterm that will require you to apply the material in an unfamiliar setting. To aid in preparation, there is such a problem on this sample exam.

How to use the solutions:

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

Math 1172

Sample Midterm 2

Spring 2016

Form A

Name: {Solutions}

OSU Username (name.nn): ______

Lecturer: ______

Recitation Instructor: ______

Recitation Time: ______

Instructions

- You have **55 minutes** to complete this exam. It consists of 6 problems on 10 pages including this cover sheet. Page 10 may be used for extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	12	
2	24	
3	16	
4	18	
5	30	
Total	100	

1. **Multiple Choice** [12 pts]

   Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

   I. The radius of convergence for the series: $\sum_{k=0}^{\infty} \frac{(2x)^k}{k!}$ is:

   A. 0 B. 1 $\boxed{\text{C. Infinite}}$ D. None of the above

   $\sum_{k=0}^{\infty} \frac{(2x)^k}{k!} = e^{2x}$, and since the ROC for e^x is infinite, so too is the ROC for e^{2x}

   II. Which of the following series is equivalent to $\sum_{k=1}^{\infty} \frac{2}{k^2+1}$?

   $\boxed{\text{A. } \sum_{k=0}^{\infty} \frac{2}{k^2+2k+2}}$ B. $\sum_{k=0}^{\infty} \frac{2}{(k-1)^2+1}$

   C. $\sum_{k=2}^{\infty} \frac{2}{(k+1)+1}$ D. None of the above

   Let $m = k-1$ (so $k=1 \leftrightarrow m=0$)
   $k = m+1$
   and $\sum_{k=1}^{\infty} \frac{2}{k^2+1} = \sum_{m+1=1}^{\infty} \frac{2}{(m+1)^2+1} = \sum_{m=0}^{\infty} \frac{2}{m^2+2m+2}$

   III. What is the appropriate form for the partial fraction decomposition for the function:

   $$f(x) = \frac{2x^2+4}{x^5+2x^3+x}?$$

   A. $f(x) = \frac{A}{x} + \frac{B}{x^4+2x^2+1}$ B. $f(x) = \frac{A}{x} + \frac{Bx+C}{(x^2+1)^2}$

   C. $f(x) = \frac{A}{x} + \frac{B}{(x+1)^2} + \frac{C}{x+1} + \frac{D}{(x-1)^2} + \frac{E}{x-1}$ $\boxed{\text{D. None of the above}}$

   $$\frac{2x^2+4}{x^5+2x^3+x} = \frac{2x^2+4}{x(x^4+2x^2+1)} = \frac{2x^2+4}{x(x^2+1)^2}$$

   $$= \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

   Irreducible!

2. **Multiselect** [18 pts]

Circle *all* of the responses that correctly answer each question. Note that there may be more than one correct response to each question or even no correct responses! Each question is worth 6 points, and in each question, you will be penalized 2 points for each incorrect response. You cannot score below a 0 for any problem here.

I. [6 pts] Circle each of the improper integrals below.

vert. asymptotes at $x=-2,-1$

A. $\boxed{\int_0^1 \frac{1}{x^2-2x}\,dx}$ B. $\int_0^6 \frac{x-3}{x^2+3x+2}\,dx$ C. $\int_{-1}^1 \frac{2x+1}{x^2+1}\,dx$

No vert asymp.

vert asymptote at $x=0$.

D. $\int_0^1 \ln|4x-9|\,dx$ E. $\boxed{\int_1^\infty \frac{2x}{5x^2+4x-2}\,dx}$ F. $\boxed{\int_0^2 \tan x\,dx}$

vert asymp at $x = 9/4$

infinite limit

vert asymp at $x = \frac{\pi}{2} < 2$.

II. [6 pts] For a given sequence $\{a_n\}_{n=1}$, define $s_n := \sum_{k=1}^{n} a_k$.

For the sequence $a_n = \frac{4}{n}$, the sequence $\{s_n\}_{n=1}$ is:

A. $\boxed{\text{Increasing}}$ B. Decreasing C. $\boxed{\text{Monotonic}}$

D. $\boxed{\text{Bounded Below}}$ E. Bounded Above F. A sequence that has a limit

- $a_n > 0$ for all n, so $s_{n+1} = s_n + a_{n+1} > s_n$. Hence, s_n is increasing. Thus, it is monotonic as well
- Since s_n is increasing it is bounded below by s_1
- $s_n = \sum_{k=1}^{n} \frac{4}{k}$, so s_n is not bdd above and has no limit since $\sum_{k=1}^{\infty} \frac{4}{k}$ diverges!

III. [6 pts] For a given sequence $\{a_n\}_{n=1}$, define $s_n := \sum_{k=1}^{n} a_k$.

Suppose that $\sum_{n=1}^{\infty} a_n = 2$. Circle each of the following statements that *MUST* be true.

A. $\lim_{n\to\infty} a_n = 2$

$\boxed{\text{B. } \lim_{n\to\infty} a_n = 0}$

C. $\lim_{n\to\infty} a_n$ does not exist

$\boxed{\text{D. } \lim_{n\to\infty} s_n = 2}$

E. $\lim_{n\to\infty} s_n = 0$

F. $\lim_{n\to\infty} s_n$ does not exist

$\boxed{\text{G. } \sum_{n=1}^{\infty} 2e^{a_n} \text{ diverges}}$

H. $\sum_{n=1}^{\infty} 1000a_n$ diverges

$\boxed{\text{I. } \sum_{n=1}^{\infty} s_n \text{ diverges.}}$

<see back of page>.

IV. [6 pts] The Taylor series for a certain function $f(x)$ centered at $x = 0$ is:

$$f(x) = 2 - 4x + 6x^3 + 3x^4 + \ldots$$

CIRCLE *all* of the following statements that *MUST* be true.

$\boxed{\text{A. } f(0) = 2}$ ← Plug in $x=0$!

$\boxed{\text{B. } f'(0) = -4}$

$f'(x) = -4 + 18x^2 + 12x^3 + \ldots$

$f'(0) = -4$.

→ C. $f'''(0) = 6$

$\boxed{\text{D. } f(2x) = 2 - 8x + 48x^3 + 48x^4 + \ldots}$.

$f(2x) = 2 - 4(2x) + 6(2x)^3 + 3(2x)^4$

$= 2 - 8x + 48x^3 + 48x^4 + \ldots$

$\boxed{\text{E. The series must converge for } x = 0.}$

F. The series must converge for $x = 1/2$.

$f''(x) = 36x + 36x^2 + \ldots$

$f'''(x) = 36 + 72x + \ldots$

→ $f'''(0) = 36$

E, F: E: The series always converges at its center

F: Cannot be determined!

III.

<u>A, B, C:</u> $\sum_{n=1}^{\infty} a_n = 2$ so $\sum_{n=1}^{\infty} a_n$ converges. Hence, $\boxed{\lim_{n\to\infty} a_n = 0}$

(or $\sum a_n$ would diverge by divergence test.

<u>D, E, F:</u> By definition, $\sum_{n=1}^{\infty} a_n$ converges iff $\lim_{n\to\infty} s_n$

and in this case, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$. Thus,

$$\boxed{\lim_{n\to\infty} s_n = 2}$$

G. Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$ so $\lim_{n\to\infty} 2e^{a_n} = 2e^0 = 2 \neq 0$.

Thus, $\boxed{\sum 2e^{a_n} \text{ diverges}}$ by divergence test.

H. $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \sum_{n=1}^{\infty} 1000\, a_n = 1000 \sum_{n=1}^{\infty} a_n$ converges.

I. $\lim_{n\to\infty} s_n = 2 \neq 0$ so $\boxed{\sum_{k=1}^{\infty} s_k \text{ diverges}}$ by div. test!

3. **Short Answer** [16 pts]

Determine whether the following statements are **True** or **False** and briefly explain your response.

I. *The function* $\frac{1}{\sqrt{x}}$ *is unbounded at* $x = 0$*, so the improper integral* $\int_0^2 \frac{1}{\sqrt{x}}\,dx$ *diverges.*

False; $\int_0^2 \frac{1}{\sqrt{x}}\,dx$ diverges iff $\lim_{a \to 0^+} \int_a^2 \frac{1}{\sqrt{x}}\,dx$ DNE, but

$$\lim_{a\to 0^+} \int_a^2 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^2 x^{-1/2}\,dx = \lim_{a\to 0^+} \left[2x^{1/2} \right|_a^2 = 2\sqrt{2}.$$

II. Suppose $\sum_{k=1}^{\infty} a_k = 3$ and $b_n = 2$ for all $n \geq 1$. Then:

A. $\sum_{k=1}^{\infty} (a_k - 3) = 0$.

False; since $\sum_{k=1}^{\infty} a_k$ converges, $\lim_{k\to\infty} a_k = 0$ so

$\lim_{k\to\infty} (a_k - 3) = -3 \neq 0$. Hence, $\sum_{k=1}^{\infty} (a_k - 3)$ diverges by divergence test.

B. $\sum_{k=1}^{\infty} (b_k - 2) = 0$.

True; let $s_n = \sum_{k=1}^{n} (b_k - 2)$. Since $b_k = 2$ for all k, $b_k - 2 = 0$ for all k, so $s_n = \sum_{k=1}^{n} 0 = 0$. Since this holds for all n, $\lim_{n\to\infty} s_n = 0 \Rightarrow \sum_{k=1}^{\infty} (b_k - 2) = 0$

C. $\sum_{k=1}^{\infty} (a_k + b_k) = 5$.

False; $\lim_{k\to\infty} (a_k + b_k) = \lim_{k\to\infty} a_k + \lim_{k\to\infty} b_k$ (since both limits exist)

$$= 0 + 2$$
$$\neq 0$$

Thus, $\sum_{k=1}^{\infty} (a_k + b_k)$ diverges by divergence test.

4. [18 pts] Determine if the following series converge or diverge. Justify your response!

I. $\sum_{k=1}^{\infty} \frac{5^{k^2}}{(2k)!}$. ← Use ratio test.

$$L = \lim_{k\to\infty} \frac{5^{(k+1)^2}}{[2(k+1)]!} \cdot \frac{(2k)!}{5^{k^2}}$$

$$= \lim_{k\to\infty} \frac{5^{k^2+2k+1}}{5^{k^2}} \cdot \frac{(2k)!}{(2k+2)!}$$

$$= \lim_{k\to\infty} \frac{\cancel{5^{k^2}}\,5^{2k+1}}{\cancel{5^{k^2}}} \cdot \frac{\cancel{(2k)!}}{(2k+2)(2k+1)\cancel{(2k)!}}$$

$$= \lim_{k\to\infty} \frac{5^{2k+1}}{(2k+2)(2k+1)}$$

$= \infty$ by growth rates → $\boxed{\sum_{k=1}^{\infty} \frac{5^{k^2}}{(2k)!} \text{ diverges by ratio test}}$

II. $\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right)$.

Note: $\ln\left(\frac{k}{k+1}\right) = \ln k - \ln(k+1)$.

Letting $a_k = \ln k - \ln(k+1)$ and $s_n = \sum_{k=1}^{n} a_k$:

$a_1 = \ln 1 - \ln 2$

$a_2 = \ln 2 - \ln 3$

$a_3 = \ln 3 - \ln 4$

$a_4 = \ln 4 - \ln 5$

$\vdots$

$a_n = \ln n - \ln(n+1)$

Telescoping series!

$s_n = \underbrace{\ln 1}_{=0} - \ln(n+1)$

Thus, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} -\ln(n+1) = -\infty \Rightarrow$ $\boxed{\sum_{k=1}^{\infty} \ln\left(\frac{k}{k+1}\right) \text{ diverges since } \lim_{n\to\infty} s_n \text{ DNE}}$

5. (Taylor Series) [30 pts]

I. Find the first four nonzero terms in the Taylor series centered at $x = 1$ for the function

$$f(x) = \sqrt{3x-2}. = (3x-2)^{1/2}$$

Computing:

$$f'(x) = \tfrac{1}{2}(3x-2)^{-1/2} \cdot 3 = \tfrac{3}{2}(3x-2)^{-1/2}$$

$$f''(x) = -\tfrac{3}{4}(3x-2)^{-3/2} \cdot 3 = -\tfrac{9}{4}(3x-2)^{-3/2}$$

$$f'''(x) = \tfrac{27}{8}(3x-2)^{-5/2} \cdot 3 = \tfrac{81}{8}(3x-2)^{-5/2}.$$

So :

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$a_n = \frac{f^{(n)}(1)}{n!}$
0	$(3x-2)^{1/2}$	1	$a_0 = 1/0! = 1$
1	$\frac{3}{2}(3x-2)^{-1/2}$	$3/2$	$a_1 = \frac{3/2}{1!} = 3/2$
2	$-\frac{9}{4}(3x-2)^{-3/2}$	$-9/4$	$a_2 = \frac{-9/4}{2!} = -\frac{9}{8}$
3	$\frac{81}{8}(3x-2)^{-5/2}$	$81/8$	$a_3 = \frac{81/8}{3!} = \frac{27}{16}$

So, the Taylor series is: $a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 = \boxed{1 + \frac{3}{2}(x-1) - \frac{9}{8}(x-1)^2 + \frac{27}{16}(x-1)^3 + \dots}$

II. Find the Taylor Series centered at $x = 0$ in summation notation for the function:

$$f(x) = \frac{3}{2+4x}.$$

Indicate its radius of convergence.

We know $\frac{1}{1-u} = \sum_{k=0}^{\infty} u$ for $|u|<1$. We need to write $f(x)$ in this form!

$$f(x) = \frac{3}{2+4x} = \frac{3}{2} \cdot \frac{1}{1+2x} = \frac{3}{2} \cdot \frac{1}{1-(-2x)}$$

$$= \frac{3}{2} \sum_{k=0}^{\infty} (-2x)^k, \quad |-2x| < 1$$

$$= \boxed{\frac{3}{2} \sum_{k=0}^{\infty} (-1)^k 2^k x^k, \quad |x| < \frac{1}{2}. \quad (\text{ROC is } \tfrac{1}{2})}$$

III. The Taylor series centered at $x = 0$ for $\ln(1+x)$ is given by:

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}, \; |x| < 1$$

Using this, find the Taylor series centered at $x = 0$ for the function:

$$y = \int x^3 \ln(1+4x^2)\,dx, \qquad y(0) = 4$$

and give its radius of convergence.

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad |x| < 1$$

so $$\ln(1+4x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(4x^2)^n}{n}, \quad |4x^2| < 1 \Rightarrow |x^2| < 1/4 \Rightarrow |x| < \frac{1}{2}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4^n x^{2n}}{n},$$

$$x^3 \ln(1+4x^2) = x^3 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4^n}{n} x^{2n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4^n}{n} x^{2n+3}, \quad |x| < \frac{1}{2} \quad (\text{ROC is } \tfrac{1}{2})$$

Hence, $$\int x^3 \ln(1+4x^2)\,dx = \int \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4^n}{n} x^{2n+3}\,dx$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4^n}{n} \int x^{2n+3}\,dx$$ term by term integration

$$= \left[\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4^n}{n} \cdot \frac{1}{2n+4}\, x^{2n+4}\right] + C$$

Using $y(0) = 4$ gives: $\underline{C = 4}$. Hence, the Taylor series for y is:

$$\boxed{4 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{4^n}{2n(n+2)} x^{2n+4}, \quad \text{ROC}: \frac{1}{2}}$$ ← since integration does not change ROC!