------------------ **DISCLAIMER** ------------------

General Information:

This midterm is a *sample* midterm. This means:

- The sample midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to the actual midterm. Note that this may be a departure from the format used on exams in previous semesters!
- The sample midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take the sample exam:

The sample midterm should be treated like the actual exam. This means:

- "Practice like you play." Schedule 55 uninterrupted minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.
- The problems on this exam are mostly based on the Worksheets posted on the Math 1152 website and your previous quizzes.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and take-home quizzes with this intent, and there will be a problem on your midterm that will require you to apply the material in an unfamiliar setting. To aid in preparation, there is such a problem on this sample exam.

How to use the solutions:

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

[redacted]
including this cover sheet. Page 10 may be used for extra workspace.

- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	24	
2	12	
3	20	
4	20	
5	24	
Total	100	

1. **Multiselect** [24 pts]

Circle *all* of the responses that correctly answer each question. Note that there may be more than one correct response to each question or even no correct responses! Each question is worth 6 points, and in each question, you will be penalized 2 points for each incorrect response. You cannot score below a 0 for any problem here.

I. [6 pts] Circle each of the improper integrals below.

only unbounded at $x=-1$, $x=-2$

continuous everywhere

unbounded at $x=0$ → $\boxed{\text{A. } \int_0^1 \frac{1}{x^2-2x}\,dx}$ B. $\int_0^6 \frac{x-3}{x^2+3x+2}\,dx$ C. $\int_{-1}^1 \frac{2x+1}{x^2+1}\,dx$

only unbounded at $x=\frac{9}{4}$ → D. $\int_0^1 \ln|4x-9|\,dx$ $\boxed{\text{E. } \int_1^\infty \frac{2x}{5x^2+4x-2}\,dx}$ $\boxed{\text{F. } \int_0^2 \tan x\,dx}$

limit is infinite

unbounded at $x=\frac{\pi}{2}<2$.

II. [6 pts] For a given sequence $\{a_n\}_{n=1}$, define $s_n := \sum_{k=1}^{n} a_k$.

For the sequence $a_n = \frac{4}{n}$, the sequence $\{s_n\}_{n=1}$ is:

$\boxed{\text{A. Increasing}}$ B. Decreasing $\boxed{\text{C. Monotonic}}$

$\boxed{\text{D. Bounded Below}}$ E. Bounded Above F. A sequence that has a limit

- $a_n > 0$ for all n. Since $s_{n+1} = s_n + a_{n+1}$.

  $s_{n+1} > s_n$ since $a_{n+1} > 0$

  So s_n is $\boxed{\text{increasing}}$. It thus is $\boxed{\text{monotonic}}$.

- Since s_n is increasing, it is $\boxed{\text{bounded below}}$ by s_1.

- Since $s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{4}{k}$, and $\sum_{k=1}^{\infty} \frac{4}{k}$ diverges (it is the harmonic series).

  $\hookrightarrow \sum_{k=1}^{\infty} \frac{4}{k} = \infty$.

s_n is neither bounded above or has a limit.

III. [6 pts] For a given sequence $\{a_n\}_{n=1}$, define $s_n := \sum_{k=1}^{n} a_k$.

Suppose that $\sum_{n=1}^{\infty} a_n = 2$. Circle each of the following statements that *MUST* be true.

A. $\lim_{n\to\infty} a_n = 2$ | B. $\boxed{\lim_{n\to\infty} a_n = 0}$ | C. $\lim_{n\to\infty} a_n$ does not exist

D. $\boxed{\lim_{n\to\infty} s_n = 2}$ | E. $\lim_{n\to\infty} s_n = 0$ | F. $\lim_{n\to\infty} s_n$ does not exist

G. $\boxed{\sum_{n=1}^{\infty} 2e^{a_n} \text{ diverges}}$ | H. $\sum_{n=1}^{\infty} 1000a_n$ diverges | I. $\boxed{\sum_{n=1}^{\infty} s_n \text{ diverges.}}$

- A, B, C: $\sum_{n=1}^{\infty} a_n = 2$ so the series converges $\Rightarrow \boxed{\lim_{n\to\infty} a_n = 0}$

- D, E, F: $\sum_{n=1}^{\infty} a_n$ converges iff $\lim_{n\to\infty} s_n$ exists and in that case,
  $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n \Rightarrow \boxed{\lim_{n\to\infty} s_n = 2}$ $\rightarrow$ <See back for rest

IV. [6 pts] Circle all of the following series that converge.

A. $\boxed{\sum_{n=0}^{\infty} e^{-n}}$ | B. $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ | C. $\boxed{\sum_{n=2}^{\infty} \frac{1}{n^2}}$

D. $\sum_{n=1}^{\infty} \arctan n$ | E. $\boxed{\sum_{n=5}^{\infty} \left[\frac{2}{n+2} - \frac{2}{n+3}\right]}$ | F. $\sum_{n=1}^{\infty} \left[\left(\frac{2}{3}\right)^n - \left(\frac{4}{7}\right)^2\right]$

A. $\sum_{n=0}^{\infty} e^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$ converges as a geometric series w/ $|r|<1$.

B. $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series w/ $p=\frac{1}{2}$, so it diverges.

C. $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a p-series w/ $p=2$ so it converges.

D. $\lim_{n\to\infty} \arctan n = \frac{\pi}{2}$, so $\sum_{n=1}^{\infty} \arctan n$ diverges by div. test.

E. $\sum_{n=5}^{\infty} \left(\frac{2}{n+2} - \frac{2}{n+3}\right)$ is a convergent telescoping series

III. G. Since $\sum_{k=1}^{\infty} a_k$ converges, $\lim_{k\to\infty} a_k = 0$ so $\lim_{k\to\infty} 2e^{a_k} = 2 \neq 0$

Hence, $\boxed{\sum_{k=1}^{\infty} 2e^{a_k} \text{ diverges}}$

H. $\sum_{k=1}^{\infty} a_k$ converges so $\sum_{k=1}^{\infty} 1000a_k = 1000\sum_{k=1}^{\infty} a_k$ converges

I. $\lim_{n\to\infty} s_n = 2$ so $\sum_{k=1}^{\infty} s_k$ diverges by divergence test.

IV. $\sum_{n=1}^{\infty}\left[\left(\frac{2}{3}\right)^n - \left(\frac{4}{7}\right)^2\right]$ diverges since $\lim_{n\to\infty}\left[\left(\frac{2}{3}\right)^n - \left(\frac{4}{7}\right)^2\right] = -\left(\frac{4}{7}\right)^2 \neq 0$.

2. **Short Answer** [12 pts]

Suppose that $\{a_n\}_{n\geq 1}$ is a sequence such that $s_n = \dfrac{15n}{4n-3}$, where $s_n = \sum_{k=1}^{n} a_k$ for all $n \geq 1$.

Provide a short response to the following questions. There is no partial credit and no penalty for guessing.

I. $a_1 + a_2 + a_3 =$ 5. ← This is just s_3! $s_3 = \frac{15(3)}{4(3)-3} = \frac{45}{9} = 5$

II. $a_3 + a_4 =$ $-\frac{18}{13}$. ← This is $s_4 - s_2$ since $s_4 = a_1 + a_2 + a_3 + a_4$
$-(s_2 = a_1 + a_2)$
$s_4 - s_2 = a_3 + a_4$

III. Determine whether $\lim_{n\to\infty} s_n$ exists. If it does, give its value.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{15n}{4n-3} = \boxed{\frac{15}{4}}$$

IV. Determine whether $\lim_{n\to\infty} a_n$ exists. If it does, give its value.

$$\lim_{n\to\infty} s_n \text{ exists} \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges} \Rightarrow \boxed{\lim_{n\to\infty} a_n = 0}$$

V. Determine whether $\sum_{k=1}^{\infty} a_k$ converges or diverges. If it converges, find the value to which it converges, or state that there is not enough information to determine this.

Since $\lim_{n\to\infty} s_n$ exists, $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$

Hence, $\boxed{\sum_{k=\infty}^{\infty} a_k \text{ converges to } \frac{15}{4}}$

VI. Determine whether $\sum_{k=1}^{\infty} s_k$ converges or diverges. If it converges, find the value to which it converges, or state that there is not enough information to determine this.

$\lim_{n\to\infty} s_n = \frac{15}{4}$, so $\boxed{\sum_{k=1}^{\infty} s_k \text{ diverges by divergence test}}$

3. I. [15 pts] Determine whether the improper integral:

$$\int_1^\infty \frac{3}{2x^2+3x}\,dx$$

converges or diverges. If it converges, find the value to which it converges.

$$\frac{3}{2x^2+3x} = \frac{3}{x(2x+3)} = \frac{A}{x} + \frac{B}{2x+3}$$

$$3 = A(2x+3) + Bx$$

$\underline{x=0:}$ $3 = 3A \to \underline{A=1}$

$\underline{x=-\frac{3}{2}:}$ $3 = -\frac{3}{2}B \to \underline{B=-2}$.

$$\text{So: } \int_1^\infty \frac{3}{2x^2+3x}\,dx = \lim_{b\to\infty} \int_1^b \left[\frac{1}{x} - \frac{2}{2x+3}\right] dx$$

Note $\int \frac{2}{2x+3}\,dx \neq 2\ln|2x+3|+C$!

$$= \lim_{b\to\infty} \left[\ln x - \ln|2x+3|\right]_1^b$$

$$= \lim_{b\to\infty} \ln \frac{x}{2x+3}\Big|_1^b$$

$$= \lim_{b\to\infty} \ln \frac{b}{2b+3} - \ln\frac{1}{5}$$

$$= \ln\frac{1}{2} - \ln\frac{1}{5}$$

$$= \boxed{\ln\frac{5}{2}}$$

II. [5 pts] Determine if the series $\sum_{k=1}^{\infty} \frac{3}{2k^2+3k}$ converges or diverges. Justify your answer!

$f(x) = \frac{3}{2x^2+3x}$ is continuous, decreasing, and nonnegative.

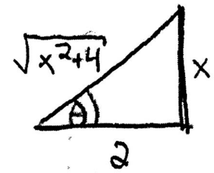
Since $\int_1^\infty f(x)\,dx$ converges, $\sum_{k=1}^{\infty} \frac{3}{2k^2+3k}$ converges by the integral test.

4. [20 pts] Find the following antiderivatives.

I. $\displaystyle\int \frac{2x+16}{(x^2+4)^2}\,dx \;=\; \int \frac{2x}{(x^2+4)^2}\,dx + \int \frac{16}{(x^2+4)^2}\,dx.$

- A u-substitution shows $\displaystyle\int \frac{2x}{(x^2+4)^2}\,dx = -\frac{1}{x^2+4} + C.$

- For $\displaystyle\int \frac{16}{(x^2+4)^2}\,dx$, let $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$ so:



$x = 2\tan\theta$

$\tan\theta = \frac{x}{2}$

$$\int \frac{16}{(x^2+4)^2}\,dx = \int \frac{16}{(4\tan^2\theta+4)^2}\cdot 2\sec^2\theta\,d\theta$$

$$= \int \frac{\cancel{16}}{\cancel{16}\sec^4\theta}\cdot 2\sec^2\theta\,d\theta$$

$$= \int 2\cos^2\theta\,d\theta \qquad \cos^2\theta = \tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta$$

$$= \int 2\left[\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right]d\theta$$

$$= \theta + \tfrac{1}{2}\sin 2\theta + C \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \theta + \sin\theta\cos\theta + C$$

$$= \arctan\frac{x}{2} + \frac{x}{\sqrt{x^2+4}}\cdot\frac{2}{\sqrt{x^2+4}} + C$$

So: $\boxed{\displaystyle\int \frac{2x+16}{(x^2+4)^2}\,dx = -\frac{1}{x^2+4} + \frac{2x}{x^2+4} + \arctan\frac{x}{2} + C}$

II. $\displaystyle\int \frac{x}{\sqrt{4-9x^2}}\,dx$ ← This is a u-sub!

$u = 4-9x^2$

$du = -18x\,dx$

$\frac{du}{-18x} = dx$

So $\displaystyle\int \frac{x}{\sqrt{4-9x^2}}\,dx = \int \frac{\cancel{x}}{\sqrt{u}}\cdot\frac{du}{-18\cancel{x}} = -\frac{1}{18}\int u^{-1/2}\,du$

$$= -\frac{1}{18}\cdot 2u^{1/2} + C$$

$$= \boxed{-\frac{1}{9}\sqrt{4-9x^2} + C}$$

5. [24 pts]

I. Consider the series:

$$\sum_{k=1}^{\infty} \frac{k^2}{k^3+1}.$$

A. CIRCLE *all* of the following tests that could be applied to test for convergence for this series. The tests only need to be applicable, *NOT* conclusive!

i) Divergence Test Always can be applied →

ii) $\frac{x^2}{x^3+1}$ is positive, decreasing and cts

iii-vi) $\frac{k^2}{k^3+1}$ is positive!

$\boxed{\text{i. Divergence Test}}$	$\boxed{\text{ii. Integral Test}}$	$\boxed{\text{iii. Ratio Test}}$
$\boxed{\text{iv. Root Test}}$	$\boxed{\text{v. Comparison Test}}$	$\boxed{\text{vi. Limit Comparison Test}}$
vii. Alternating Series Test	viii. Geometric Series Test	ix. p-series Test

series is not alternating!

This is not a geometric series or a p-series

B. Use one of the applicable tests above to determine if the series converges or diverges. If the series converges, determine whether the series converges absolutely or conditionally.

- The summand is a rational expression in k, so use Limit comparison test. Note $\frac{k^2}{k^3+1} \sim \frac{k^2}{k^3} = \frac{1}{k}$, so we compare with $\sum \frac{1}{k}$.
- Since $\lim_{k\to\infty} \left(\frac{k^2}{k^3+1}\right) / \left(\frac{1}{k}\right) = \lim_{k\to\infty} \frac{k^2}{k^3+1} \cdot \frac{k}{1} = \lim_{k\to\infty} \frac{k^3}{k^3+1} = 1$

  is non-zero and finite, the Limit Comparison test ensures $\sum \frac{k^2}{k^3+1}$ and $\sum \frac{1}{k}$ will both either converge or diverge.
- Since $\sum \frac{1}{k}$ is the harmonic series, it diverges
- Hence $\sum_{k=1}^{\infty} \frac{k^2}{k^3+1}$ diverges as well.

II. Consider the series:

$$\sum_{k=1}^{\infty}(-1)^k \frac{k^2}{k^3+1}.$$

A. CIRCLE *all* of the following tests that could be applied to test for convergence for this series. The tests only need to be applicable, *NOT* conclusive!

ii-vi) fail because the summand is not positive!

$\boxed{\text{i. Divergence Test}}$	ii. Integral Test	iii. Ratio Test
iv. Root Test	v. Comparison Test	vi. Limit Comparison Test
$\boxed{\text{vii. Alternating Series Test}}$	viii. Geometric Series Test	ix. p-series Test

B. Determine if the series converges or diverges. If the series converges, determine whether the series converges absolutely or conditionally.

- We determined $\sum_{k=1}^{\infty}\left|(-1)^k \frac{k^2}{k^3+1}\right| = \sum \frac{k^2}{k^3+1}$ diverges, so the series does not converge absolutely.

- Note that the series is alternating and:
  1) $\frac{k^2}{k^3+1}$ is decreasing
  2) $\lim_{k\to\infty} \frac{k^2}{k^3+1} = 0$

so the series converges by the Alternating Series Test.

Hence, $\sum_{k=1}^{\infty}(-1)^k \frac{k^2}{k^3+1}$ converges conditionally