DISCLAIMER

General Information:

This midterm is a *sample* midterm. This means:

- The sample midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to the actual midterm. Note that this may be a departure from the format used on exams in previous semesters!
- The sample midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take the sample exam:

The sample midterm should be treated like the actual exam. This means:

- "Practice like you play." Schedule 55 uninterrupted minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.
- The problems on this exam are mostly based on the Worksheets posted on the Math 1172 website and your previous quizzes.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and take-home quizzes with this intent, and there will be a problem on your midterm that will require you to apply the material in an unfamiliar setting. To aid in preparation, there is such a problem on this sample exam.

How to use the solutions:

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

Math 1172

Sample Midterm 2

Spring 2016

Form B

Name: Solutions

OSU Username (name.nn): ____________________

Lecturer: ____________________

Recitation Instructor: ____________________

Recitation Time: ____________________

Instructions

- You have **55 minutes** to complete this exam. It consists of 6 problems on 10 pages including this cover sheet. Page 10 may be used for extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	12	
2	20	
3	18	
4	15	
5	35	
Total	100	

1. **Multiple Choice** [12 pts]

   Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

   I. If $\sum_{k=1}^{\infty} a_k = 5$ and $\sum_{k=1}^{\infty}(a_k + b_k) = 3$, what is $\lim_{n\to\infty} b_n$?

   A. 0 B. 2 C. Does not exist

   D. This cannot be determined unless we have a formula for b_n.

   E. None of the above.

   $\sum_{k=1}^{\infty}(a_k+b_k)$ converges, so $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty}(a_k+b_k) - \sum_{k=1}^{\infty} a_k$ by properties of convergent series. Hence, $\sum_{k=1}^{\infty} b_k = -2$ so $\sum_{k=1}^{\infty} b_k$ converges $\Rightarrow$ $\boxed{\lim_{k\to\infty} b_k = 0}$

   II. Let $\sum_{k=1}^{\infty} a_k$ be a series for which $\sum_{k=1}^{\infty} s_k$ diverges, where $s_n = \sum_{k=1}^{n} a_k$. Then, $\sum_{k=1}^{\infty} a_k$:

   A. Converges to 0 B. Diverges

   C. Converges, but more information is needed to determine its value.

   D. Could converge or diverge; not enough information is given.

   Inconclusive information is given to find $\lim_{k\to\infty} s_k$!

   III. Let $f(x)$ be a continuously differentiable function whose power series is:

   $$f(x) = \sum_{k=1}^{\infty} a_k(x-4)^k.$$

   Let R be the radius of convergence for this series, and suppose it is known that the series $\sum_{k=1}^{\infty} a_k$ converges. Then:

   A. $R < 1$ B. $R = 1$ C. $R > 1$

   D. $R = \infty$; i.e. the series converges for all x.

   E. None of these.

   Note $\sum_{k=1}^{\infty} a_k = f(5)$. Since the series is centered at $x=4$, $R \geq 1$, but not enough information is given to determine more!

2. **Multiselect** [20 pts]

Circle *all* of the responses that correctly answer each question. Note that there may be more than one correct response to each question or even no correct responses! Each question is worth 10 points, and in each question, you will be penalized 2 points for each incorrect response. You cannot score below a 0 for any problem here.

I. [10 pts] Suppose that $\{a_n\}_{n\geq 1}$ and $a_n > 0$ for all $n \geq 1$. Let $s_n = \sum_{k=1}^{n} a_k$ and suppose $\lim_{n\to\infty} s_n = 2$.

A. $\boxed{\sum_{k=1}^{\infty} a_k = 2}$

B. $\boxed{\sum_{k=2}^{\infty} a_k < 2}$

C. $\boxed{\lim_{n\to\infty} a_n = 0}$

D. $\sum_{k=1}^{\infty} (a_k - 2) = 0$

E. $\boxed{\{s_n\} \text{ MUST be bounded.}}$

F. $\boxed{\{s_n\} \text{ MUST be monotonic.}}$

G. $\boxed{\sum_{k=1}^{\infty} s_k \text{ MUST diverge.}}$

H. $\boxed{\sum_{k=1}^{\infty} \ln s_k \text{ MUST diverge.}}$

I. $\boxed{\sum_{k=1000}^{\infty} a_k \text{ converges.}}$

II. [10 pts] Circle all of the following series that converge.

A. $\boxed{\sum_{k=0}^{\infty} e^{-k}}$

B. $\sum_{k=300}^{\infty} \frac{1}{k}$

C. $\sum_{k=2}^{\infty} \frac{3^{2k}}{4^k}$

D. $\sum_{k=1}^{\infty} \arctan k$

E. $\boxed{\sum_{k=5}^{\infty} \left[\frac{2}{k+2} - \frac{2}{k+3}\right]}$

F. $\sum_{k=1}^{\infty} \left[\left(\frac{2}{3}\right)^k - \left(\frac{4}{7}\right)^2\right]$

G. $\sum_{k=1}^{\infty} \frac{3k+1}{2k+2}$

H. $\sum_{k=5}^{\infty} \frac{2k+\sin k}{k+2}$

I. $\boxed{\sum_{k=1}^{\infty} \frac{3^k}{k!}}$

A. $\sum_{k=0}^{\infty} e^{-k} = \sum_{k=0}^{\infty} \left(\frac{1}{e}\right)^k \rightarrow$ converges as a geom series w/ $|r|<1$.

B. Diverges; it is a tail of the harmonic series!

C. Diverges; $\frac{3^{2k}}{4^k} = \frac{(3^2)^k}{4^k} = \left(\frac{9}{4}\right)^k \leftarrow$ geometric w/ $r>1$.

D. Diverges; $\lim_{k\to\infty} \arctan k = \frac{\pi}{2} \neq 0$.

2I. A. $\lim_{n\to\infty} s_n = 2$ so $\sum_{k=1}^{\infty} a_k = 2$ by definition. $\to$ $\boxed{\text{True}}$

B. Since $a_k > 0$ for all k, $\sum_{k=2}^{\infty} a_k < \sum_{k=1}^{\infty} a_k = 2$. $\to$ $\boxed{\text{True}}$

C. $\sum_{k=1}^{\infty} a_k$ converges, so $\lim_{n\to\infty} a_n = 0$. $\boxed{\text{True}}$

D. $\sum_{k=1}^{\infty} (a_k - 2)$ diverges. Since $\sum_{k=1} a_k$ converges, $\lim_{k\to\infty} a_k = 0$ so $\lim_{k\to\infty} (a_k - 2) = -2$. Hence, $\sum (a_k - 2)$ diverges by divergence test

E. $\boxed{\text{True}}$; since $a_n > 0$ for all n, s_n is increasing. Since $\lim_{n\to\infty} s_n$ exists, s_n must be bounded.

F. $\boxed{\text{True}}$ (see above; $a_n > 0$ for all $n \Rightarrow s_n$ is increasing!).

G. $\boxed{\text{True}}$, $\lim_{n\to\infty} s_n = 2 \neq 0$ so $\sum_{n=1}^{\infty} s_n$ diverges by div. test

H. $\boxed{\text{True}}$, $\lim_{n\to\infty} \ln s_n = \ln 2$ so $\sum \ln s_n$ diverges by div test.

I. $\boxed{\text{True}}$

II. E. Converges as a telescoping series:

F. Diverges; $\lim_{k\to\infty} \left[\left(\frac{2}{3}\right)^k - \left(\frac{4}{7}\right)^2\right] = 0 - \left(\frac{4}{7}\right)^2 \neq 0$.

G. Diverges; $\lim_{k\to\infty} \frac{3k+1}{2k+2} = \frac{3}{2} \neq 0$.

H. Diverges; $\frac{2k-1}{k+2} \leq \frac{2k + \sin k}{k+2} \leq \frac{2k+1}{k+2}$ for all k so $\frac{2k+\sin k}{k+2} \to 2$ by squeeze thm, Hence $\sum \frac{2k+\sin k}{k+2}$ diverges by div test.

I. Converges; $L = \lim \frac{3^{k+1}}{(k+1)!} \cdot \frac{k!}{3^k} = 0$ so this conv. by ratio test

3. **Short Answer** [18 pts]

Suppose that $\{a_n\}_{n\geq 1}$ is a sequence such that $s_n = \dfrac{15n}{4n-3}$, where $s_n = \displaystyle\sum_{k=1}^{n} a_k$ for $n \geq 1$.

Provide a short response to the following questions. There is no partial credit and no penalty for guessing.

I. $a_1 + a_2 + a_3 =$ $\underline{5}$.

$\leftarrow a_1 + a_2 + a_3 = s_3 = \dfrac{15(3)}{4(3)-3} = 5.$

II. $a_3 + a_4 =$ $\underline{\frac{-18}{13}}$.

$s_4 = a_1 + a_2 + a_3 + a_4$

$s_2 = a_1 + a_2$

$s_4 - s_2 = a_3 + a_4$

but $s_4 - s_2 = \dfrac{15(4)}{4(4)-3} - \dfrac{15(2)}{4(2)-3} = \dfrac{-18}{13}$.

III. Determine whether $\lim_{n\to\infty} s_n$ exists. If it does, give its value.

$$\boxed{\lim_{n\to\infty} s_n = \frac{15}{4}}$$

IV. Determine whether $\lim_{n\to\infty} a_n$ exists. If it does, give its value.

$\lim_{n\to\infty} s_n$ exists $\Rightarrow \sum_{k=1}^{\infty} a_k$ converges $\Rightarrow$ $\boxed{\lim_{k\to\infty} a_k = 0}$

V. Determine whether $\sum_{k=1}^{\infty} a_k$ converges or diverges. If it converges, find the value to which it converges, or state that there is not enough information to determine this.

$\boxed{\sum_{k=1}^{\infty} a_k \text{ converges to } \frac{15}{4}}$ since $\lim_{n\to\infty} s_n = L \Leftrightarrow \sum_{k=1}^{\infty} a_k = L$

VI. Determine whether $\sum_{k=1}^{\infty} s_k$ converges or diverges. If it converges, find the value to which it converges, or state that there is not enough information to determine this.

$\lim_{k\to\infty} s_k = \frac{15}{4} \neq 0 \Rightarrow$ $\boxed{\sum_{k=1}^{\infty} s_k \text{ diverges by div. test}}$

4. I. [5 pts] Use partial fraction decomposition to show that:

$$\frac{3x^2+2}{x(x^2+1)^2} = \frac{2}{x} - \frac{2x}{x^2+1} + \frac{x}{(x^2+1)^2}$$

$$\frac{3x^2+2}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \rightarrow \boxed{\text{Need } A=2,\ B=-2,\ C=0,\ D=1,\ E=0}$$

$$3x^2+2 = (x^2+1)^2 A + x(x^2+1)(Bx+C) + x(Dx+E)$$

$\underline{x=0:}$ $\boxed{2 = A}$

So: $3x^2+2 = (x^2+1)^2 \cdot 2 + x(x^2+1)(Bx+C) + x(Dx+E)$

$$3x^2+\cancel{2} = 2x^4+4x^2+\cancel{2} + Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex$$

$\underline{\text{Collect like powers}}$ $-2x^4 - x^2 = Bx^4 + Cx^3 + (B+D)x^2 + (C+E)x$

So: comparing coeff: $\boxed{B=-2}$, $\boxed{C=0}$, $B+D=-1$, $C+E=0$

$-2+D=-1$ $\boxed{D=1}$ $0+E=0$ $\boxed{E=0}$

II. [10 pts] Determine whether the improper integral:

$$\int_1^\infty \frac{3x^2+2}{x(x^2+1)^2}\,dx$$

converges or diverges. If it converges, find the value to which it converges.

$$\int_1^\infty \frac{3x^2+2}{x(x^2+1)^2}\,dx = \lim_{b\to\infty} \int_1^b \left[\frac{2}{x} - \frac{2x}{x^2+1} + \frac{x}{(x^2+1)^2}\right]dx$$

$$= \lim_{b\to\infty} \left[2\ln|x| - \ln(x^2+1) - \frac{1}{2}\frac{1}{x^2+1}\right|_1^b$$

MAKE SURE YOU WORK OUT THE INTEGRALS HERE

$$= \lim_{b\to\infty} \left[\ln x^2 - \ln(x^2+1) - \frac{1}{2}\frac{1}{x^2+1}\right|_1^b$$

$$= \lim_{b\to\infty} \left[\ln\frac{x^2}{x^2+1} - \frac{1}{2}\frac{1}{x^2+1}\right|_1^b$$

$$= \lim_{b\to\infty} \left\{\left[\ln\frac{b^2}{b^2+1} - \frac{1}{2}\frac{1}{b^2+1}\right] - \left[\ln\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2}\right]\right\}$$

$$= \ln 1 - 0 - \ln\frac{1}{2} + \frac{1}{4}$$

$$= \boxed{\ln 2 + \frac{1}{4}}$$

5. [35 pts] (Taylor Series) MAKE SURE YOU KNOW WHAT THIS MEANS!

I. [7 pts] Find the first 4 nonzero terms in the Taylor series centered at $x = 0$ for the function:

$$f(x) = e^{-x^3} + x\sin(x^2).$$

This would be a pain to differentiate! Instead:

- $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$

$e^{-x^3} = 1 + (-x^3) + \frac{1}{2}(-x^3)^2 + \frac{1}{6}(-x^3)^3 + \ldots$

$e^{-x^3} = 1 - x^3 + \frac{1}{2}x^6 - \frac{1}{6}x^9 + \ldots$ ← Series is valid to x^9 term!

- $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots$

$x\sin(x^2) = x\left[(x^2) - \frac{1}{3!}(x^2)^3 + \frac{1}{5!}(x^2)^5 - \ldots\right]$

$x\sin x^2 = x^3 - \frac{1}{6}x^7 + \frac{1}{120}x^{11} - \ldots$ ← series is valid to x^{11} term.

When we add the series, the result is valid up to x^9. If we need more terms, we'll have to write out more in e^{-x^3}!

Add term by term:

$$\boxed{e^{-x^3} + x\sin x^2 = 1 + \frac{1}{2}x^6 - \frac{1}{6}x^7 - \frac{1}{6}x^9 + \ldots}$$

II. [10 pts] Compute $\lim_{x\to 0} \frac{x^4 e^{x^2}}{2\cos(x^2) - 2 + x^2}$.

$e^x = 1 + x + \frac{1}{2!}x^2 + \ldots$

$x^4 e^{x^2} = x^4\left[1 + (x^2) + \frac{1}{2!}(x^2)^2 + \ldots\right]$

$x^4 e^{x^2} = x^4 + x^6 + \frac{1}{2}x^8 + \ldots$

$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots$

$2\cos x^2 = 2\left[1 - \frac{1}{2!}(x^2)^2 + \frac{1}{4!}(x^2)^4 - \ldots\right]$

$2\cos x^2 = 2 - x^4 + \frac{1}{12}x^8 - \ldots$

So $2\cos x^2 - 2 + x^2 = x^2 - x^4 + \frac{1}{12}x^8 - \ldots$

Thus, $\lim_{x\to 0} \frac{x^4 e^{x^2}}{2\cos(x^2) - 2 + x^2}$

$= \lim_{x\to 0} \frac{x^4 + x^6 + \frac{1}{2}x^8 + \ldots}{x^2 - x^4 + \frac{1}{12}x^8 + \ldots}$

$= \lim_{x\to 0} \frac{x^4(1 + x^2 + \ldots)}{x^2(1 - x^2 + \ldots)}$

$= \lim_{x\to 0} x^2\left(\frac{1 + x^2 + \ldots}{1 - x^2 + \ldots}\right)$

$= \boxed{0}$

III. [18 pts] Given the power series $f(x) = \sum_{k=1}^{\infty} \frac{k(x+2)^{2k}}{4^k}$:

A. [1 pt] State the center of the series.

$\boxed{\text{Center is } -2}$

↳ $\sum a_k (x-c)^k$ is centered at $x=c$.

"The Taylor series is an "infinite" polynomial in powers of $x-c$."

B. [6 pts] Find the radius of convergence for the power series.

Use the Ratio Test.

$$L(x) = \lim_{k\to\infty} \left| \frac{(k+1)(x+2)^{2(k+1)}}{4^{k+1}} \cdot \frac{4^k}{k(x+2)^{2k}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{k+1}{k} \cdot \frac{(x+2)^{2k+2}}{(x+2)^{2k}} \cdot \frac{4^k}{4^{k+1}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{k+1}{k} \cdot \frac{\cancel{(x+2)^{2k}}(x+2)^2}{\cancel{(x+2)^{2k}}} \cdot \frac{\cancel{4^k}}{\cancel{4^k}4^1} \right|$$

$$= \frac{1}{4}|x+2|^2 \lim_{k\to\infty} \frac{k+1}{k}$$

$$L(x) = \frac{1}{4}|x+2|^2$$

The ratio test shows that the series will converge for all x-values that make $L(x) < 1$ so set $L(x) < 1$:

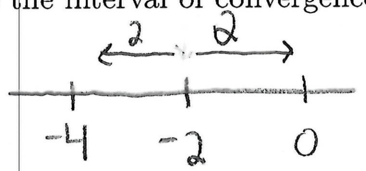
$$\frac{1}{4}|x+2|^2 < 1$$

$$|x+2|^2 < 4$$

$$|x+2| < \boxed{2.}$$

$\boxed{\text{The radius of convergence is } 2}$

C. [3 pts] State the interval of convergence for the power series.



$\rightarrow$ $\boxed{\text{Interval is } (-4,0) \text{ or } -4<x<0}$

D. [6 pts] Find a power series representation for $g(x) = f'(x)$ and give its radius of convergence.

$$f(x) = \sum_{k=1}^{\infty} \frac{k(x+2)^{2k}}{4^k}$$

$$f'(x) = \frac{d}{dx}\left[\sum_{k=1}^{\infty} \frac{k(x+2)^{2k}}{4^k}\right]$$

$$= \sum_{k=1}^{\infty} \frac{d}{dx}\left[\frac{k}{4^k}\cdot(x+2)^{2k}\right]$$

"Term by term" differentiate

$$= \sum_{k=1}^{\infty} \frac{k}{4^k} \frac{d}{dx}(x+2)^{2k}$$

$$= \sum_{k=1}^{\infty} \frac{k}{4^k}\cdot 2k(x+2)^{2k-1} = \boxed{\sum_{k=1}^{\infty} \frac{2k^2}{4^k}(x+2)^{2k-1}}$$

The ROC does NOT change when we differentiate, so the ROC for this series is 2

E. [2 pts] Find $g(-2), g'(-2), g''(-2)$ and $g'''(-2)$.

Write out several terms:

$g(x) = \frac{1}{2}(x+2) + \frac{1}{2}(x+2)^3 + \frac{9}{32}(x+2)^5 + \ldots \rightarrow \boxed{g(-2)=0}$

$g'(x) = \frac{1}{2} + \frac{3}{2}(x+2)^2 + \frac{45}{32}(x+2)^4 \rightarrow \boxed{g'(-2)=\frac{1}{2}}$

$g''(x) = 3(x+2) + \frac{45}{8}(x+2)^3 + \ldots \rightarrow \boxed{g''(-2)=0}$

$g'''(x) = 3 + \frac{135}{8}(x+2)^2 + \ldots \rightarrow \boxed{g'''(-2)=3}$

Bonus: [2 pts] Find $f^{(20)}(-2)$.

Note: $\frac{f^{(n)}(-2)}{n!} = a_n$, where a_k is the coefficient of $(x+2)^n$ (by defn of the Taylor polynomial!).

$\rightarrow \frac{f^{(20)}(-2)}{20!} = a_{20}$ or $f^{(20)}(-2) = 20!\, a_{20}$

Since $f(x) = \sum_{k=1}^{\infty} \frac{k(x+2)^{2k}}{4^k}$, $(x+2)^{2k} = (x+2)^{20}$ when $k=10$

$\rightarrow a_{20} = \frac{10}{4^{10}}$. So $\boxed{f^{(20)}(-2) = 20!\cdot\frac{10}{4^{10}}}$