------------------ DISCLAIMER ------------------

General Information:

This midterm is a *sample* midterm. This means:

- The sample midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to the actual midterm. Note that this may be a departure from the format used on exams in previous semesters!
- The sample midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take the sample exam:

The sample midterm should be treated like the actual exam. This means:

- "Practice like you play." Schedule 55 minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.
- The problems on this exam are mostly based on the Worksheets posted on the Math 1152 website and your previous quizzes.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and take-home quizzes with this intent, and there could be a problem on the exam that requires you to apply the material in an unfamiliar setting. To aid in preparation, there is such a problem on this sample exam.

How to use the solutions:

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

Math 1152

Sample Midterm 3

Autumn 2016

Form A

Name: ____________________

OSU Username (name.nn): ____________________

Lecturer: ____________________

Recitation Instructor: ____________________

Recitation Time: ____________________

Instructions

- You have **55 minutes** to complete this exam. It consists of 5 problems on 10 pages including this cover sheet.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	12	
2	10	
3	12	
4	16	
5	30	
6	20	
Total	100	

1. **Multiple Choice** [12 pts]

Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

I. The polar form of a curve in the xy plane is given by $r = \cos(2\theta)$.

Which of the following is the Cartesian description of the curve?

A. $x^2 + y^2 = x$

B. $x^2 + y^2 = 2x$

C. $(x^2 + y^2)^3 = (x^2 - y^2)^2$

D. None of the above

$r = \cos(2\theta) = \cos^2\theta - \sin^2\theta$ (from a trig identity)

$r^3 = r^2\cos^2\theta - r^2\sin^2\theta \rightarrow (x^2+y^2)^{3/2} = x^2 - y^2$

$(x^2+y^2)^3 = (x^2-y^2)^2$

II. A curve is described parametrically by:

$$\begin{cases} x(t) &= 2\cos t \\ y(t) &= 3\sin t \end{cases}, \quad -\infty \leq t \leq \infty$$

How many *distinct* vertical tangent lines does the curve have?

A. 0

B. 1

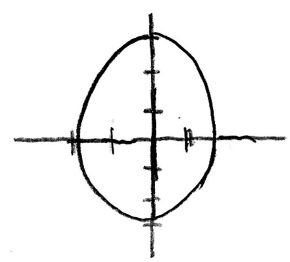
C. 2

D. More than 2

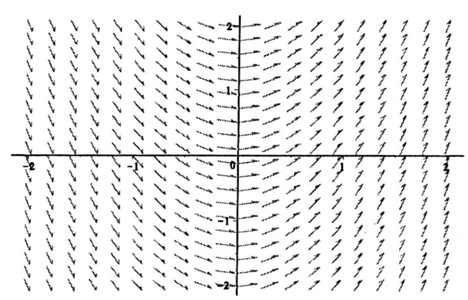
$x = 2\cos t \rightarrow \cos t = \frac{1}{2}x$

$y = 3\sin t \rightarrow \sin t = \frac{1}{3}y$

Using $\sin^2 t + \cos^2 t = 1 \rightarrow \frac{1}{4}x^2 + \frac{1}{9}y^2 = 1$



III. The direction field for a certain differential equation is shown below:



← • slopes are positive in QI → Not D

• slopes are positive in QIV → Not A, C

For which of the following differential equations could this the direction field?

A. $\frac{dy}{dx} = 2y$

B. $\frac{dy}{dx} = 2x$

C. $\frac{dy}{dx} = 2xy$

D. $\frac{dy}{dx} = -2xy$

2. **Multiselect** [10 pts]

Directions: Circle *all* of the responses that MUST be true for the problem below. Note that there may be more than one correct response or even no correct responses!

A perfect answer for this question is worth 10 points. You will be penalized 2 points for:

- each incorrect choice that you circle.
- each correct choice that you do not circle.

Thus, the possible grades on this problem are 0, 2, 4, 6, 8 or 10. You cannot score below a 0 for this problem.

Problem: The first four nonzero terms in the Taylor series for a certain function $f(x)$ centered at $x = 0$ are:

$$f(x) = x - 3x^2 + 2x^3 + x^5 + \dots$$

Suppose it is known that:

- The series for $f(1)$ converges.
- The series for $f(-2)$ diverges.

CIRCLE *all* of the following statements that *MUST* be true.

A. $f(0) = 0$.

B. $f'(0) = 1$.

C. $f''(0) = -3$

D. $f\left(\frac{x}{2}\right)$ diverges when $x = -2$.

E. The series for $f(x)$ must converge for $x = 0$.

F. The series for $f(x)$ must converge for $x = 1/2$.

G. The radius of convergence for the series for $f(x)$ is at most 2.

H. The radius of convergence for the series for $f'(x)$ is at least 1.

B. $f'(x) = 1 - 6x + 6x^2 + 5x^4 + \dots \rightarrow \underline{f'(0) = 1}$

C. $f''(x) = -6 + 12x + 20x^3 + \dots \rightarrow f''(0) = -6$.

D. When $x = -2$, $f(\frac{x}{2}) = f(-1)$. We know $f(1)$ converges but $f(-1)$ need not!

ex: $f(x) = x - 3x^2 + 2x^3 + x^5 + \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k}$ converges at 1 by AST but diverges at $x = -1$ (harmonic series).

E. Series always converge at their centers

F. The series converges at $x = 1$, so the min ROC is $1 \Rightarrow$ The series converges on $(-1, 1)$.

G. The series diverges when $x = 2$, which is 2 units from the center $\Rightarrow$ ROC is at most 2

H. Differentiating does not change the ROC.

3. (Short Answer) [12 pts]

I. Given that $f(-1) = 0$, $f'(-1) = 2$, $f''(-1) = 3$, and $f'''(-1) = 6$, write down the **second** degree Taylor polynomial of $f(x)$ centered at $x = -1$.

$P_2(x) = a_0 + a_1(x-c) + a_2(x-c)^2$, where $a_k = \frac{f^{(k)}(c)}{k!}$. Here, $c=-1$

$\Rightarrow P_2(x) = 0 + 2(x+1) + \frac{3}{2!}(x+1)^2 \rightarrow \boxed{P_2(x) = 2(x+1) + 3(x+1)^2}$

II. Find all possible values of a so $y = e^{at}$ is a solution to the differential equation:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 0.$$

$y = e^{at}$
$y' = ae^{at}$
$y'' = a^2e^{at}$

When $y = e^{at}$ is a solution:

$a^2e^{at} + 4ae^{at} + 3e^{at} = 0$

$e^{at}\left[a^2 + 4a + 3\right] = 0$

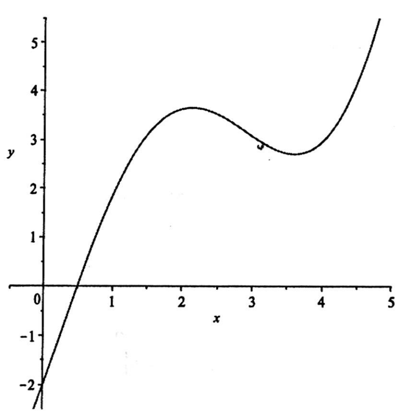
$(a+3)(a+1) = 0 \Rightarrow \boxed{a=-1 \text{ or } a=-3}$

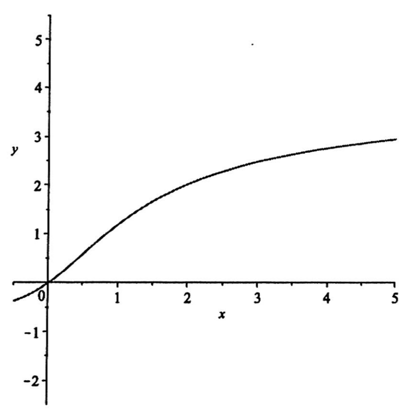
III. Determine whether either curve below could be a solution to the initial value problem:

$$\frac{dy}{dx} = \frac{y}{x^2+1}, \qquad y(0) = 2.$$

You only have to give 1 reason

If one of these curves does not represent a solution to this problem, explain why!





Not a solution for many reasons:

- In QI, where $x, y > 0$, we must have $y' > 0$, so the curve should increase in QI!
- Graph has extrema at $x=2$ and near $x=4$ but $y' \neq 0$ there!
- Graph should decrease in QIV because y' is negative there.

Not a solution since $y(0)=0$ here but we require $y(0)=2$ in the IVP!

4. [16 pts] The curve C is described parametrically by:

$$\begin{cases} x(t) &= \dfrac{1-4t^2}{t^4} \\ y(t) &= t^2 \end{cases}, \quad t > 0$$

I. The Tangent Line at $(x, y) = (-3, 1)$: Part I

A. Eliminate the parameter to find an implicit description of C in terms of x and y.

$$x = \frac{1-4t^2}{t^4} = \frac{1-4t^2}{(t^2)^2}$$

Since $y = t^2$, $\quad x = \dfrac{1-4y}{y^2}$

$$xy^2 = 1-4y \Rightarrow \boxed{xy^2 + 4y = 1}$$

B. Use implicit differentiation to find y' in terms of x and y only.

· Take $\frac{d}{dx}$ of both sides, remembering to use chain rule when differentiating y:

$$\underbrace{y^2 + 2xy\frac{dy}{dx}}_{\text{From product rule}} + 4\frac{dy}{dx} = 0.$$

$$2xy\frac{dy}{dx} + 4\frac{dy}{dx} = -y^2$$

$$(2xy+4)\frac{dy}{dx} = -y^2$$

$$\boxed{\frac{dy}{dx} = -\frac{y^2}{2xy+4}}$$

C. Find the equation of the tangent line to the curve when $(x, y) = (-3, 1)$. Solve for y explicitly in your final answer.

The slope is $\left.\dfrac{dy}{dx}\right|_{(x,y)=(-3,1)} = -\dfrac{(1)^2}{2(-3)(1)+4}$

$$= \frac{1}{2}.$$

So the tan line is:

$$y - 1 = \frac{1}{2}(x-(-3))$$

$$y - 1 = \frac{1}{2}x + \frac{3}{2}$$

$$\boxed{y = \frac{1}{2}x + \frac{5}{2}}$$

II. The Tangent Line at $(x, y) = (-3, 1)$: Part II

A. Find an expression for y' in terms of t only from the original parametric description of the curve.

$$\frac{dy}{dt} = 2t$$

$$\frac{dx}{dt} = \frac{-8t(t^4) - 4t^3(1-4t^2)}{(t^4)^2} = \frac{-8t^5 - 4t^3 + 16t^5}{t^8} = \frac{8t^2-4}{t^5}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{\frac{8t^2-4}{t^5}} = \boxed{\frac{2t^6}{8t^2-4}}$$

B. Find the t-value so $(x(t), y(t)) = (-3, 1)$.

When $y = 1$, $t^2 = 1$ so $t = \pm 1$. Since the curve is defined for $t > 0$, we use $\boxed{t = 1}$.

Checking $x(t)$ when $t = 1$:

$$x(1) = \frac{1 - 4(1)^2}{(1)^4} = -3 \checkmark$$

C. Find the equation of the tangent line to the curve when $(x, y) = (-3, 1)$. Solve for y explicitly in your final answer.

$$y - y(1) = m(x - x(1))$$

Letting $t = 1$, we find $m = \left.\frac{dy}{dx}\right|_{t=1} = \frac{2(1)^6}{8(1)^2 - 4} = \frac{2}{4} = \frac{1}{2}$.

So:

$$y - 1 = \frac{1}{2}(x - (-3))$$

$$\boxed{y = \frac{1}{2}x + \frac{5}{2}}$$

5. [25 pts] (Taylor Series)

I. Find the first four nonzero terms in the Taylor series centered at $x = 0$ for the function

$$f(x) = x + 6e^{x^2} - 3x^3 \sin x.$$

Simplify your final answer completely!

Hint: Using the definition to do this is *very* painful!

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

$$e^{x^2} = 1 + (x^2) + \frac{1}{2}(x^2)^2 + \frac{1}{6}(x^2)^3 + \frac{1}{24}(x^2)^4 + \dots$$

$$6e^{x^2} = 6 + 6x^2 + 3x^4 + x^6 + \frac{1}{4}x^8 + \dots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

$$3x^3 \sin x = 3x^4 - \frac{1}{2}x^6 + \frac{1}{60}x^8 - \dots$$

So up to x^8:

$$f(x) = x + \left[6 + 6x^2 + \cancel{3x^4} + x^6 + \frac{1}{4}x^8 + \dots\right]$$

$$- \left[\cancel{3x^4} - \frac{1}{2}x^6 + \frac{1}{60}x^8 + \dots\right]$$

$$= \boxed{6 + x + 6x^2 + \frac{1}{2}x^6} + \dots$$

II. A. Find the first 4 nonzero terms in the Taylor series centered at $x = 0$ for the function:

$$f(x) = \sqrt{1+x}. = (1+x)^{1/2}.$$

k	$f^{(k)}(x)$	$f^{(k)}(0)$	$a_k = \frac{f^{(k)}(0)}{k!}$
0	$(1+x)^{1/2}$	1	$1/0! = 1$
1	$\frac{1}{2}(1+x)^{-1/2}$	$1/2$	$\frac{1/2}{1!} = \frac{1}{2}$
2	$-\frac{1}{4}(1+x)^{-3/2}$	$-1/4$	$\frac{-1/4}{2!} = -\frac{1}{8}$
3	$\frac{3}{8}(1+x)^{-5/2}$	$3/8$	$\frac{3/8}{3!} = \frac{1}{16}$

$$\sqrt{1+x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= \boxed{1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots}$$

B. Use Taylor series to evaluate the limit:

$$\lim_{x\to 0} \frac{1-\sqrt{1+x^4}}{x^3 \sin(2x)}.$$

- Using A: $\sqrt{1+x^4} = 1 + \frac{1}{2}(x^4) - \frac{1}{8}(x^4)^2 + \dots$

  So $1 - \sqrt{1+x^4} = -\frac{1}{2}x^4 + \frac{1}{8}x^8 - \dots$

- $\sin x = x - \frac{1}{3!}x^3 + \dots$

  $\sin(2x) = (2x) - \frac{1}{6}(2x)^3 + \dots$

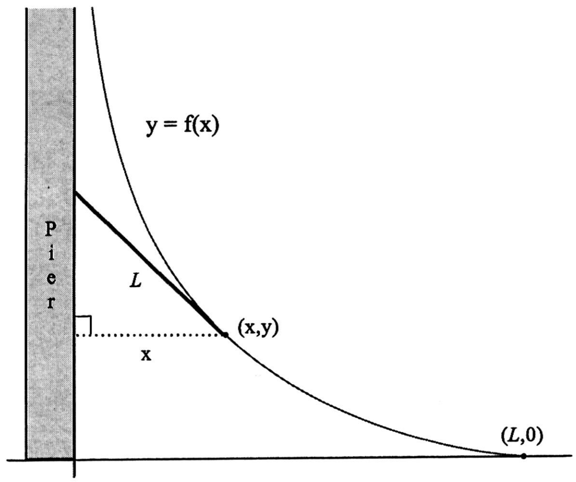
  $x^3 \sin(2x) = 2x^4 - \frac{4}{3}x^6 + \dots$

So: $$\lim_{x\to 0} \frac{1-\sqrt{1+x^4}}{x^3\sin(2x)} = \lim_{x\to 0} \frac{-\frac{1}{2}x^4 + \frac{1}{8}x^8 - \dots}{2x^4 - \frac{4}{3}x^6 + \dots}$$

$$= \lim_{x\to 0} \frac{\cancel{x^4}\left[-\frac{1}{2} + \frac{1}{8}x^4 - \dots\right]}{\cancel{x^4}\left[2 - \frac{4}{3}x^2 + \dots\right]}$$

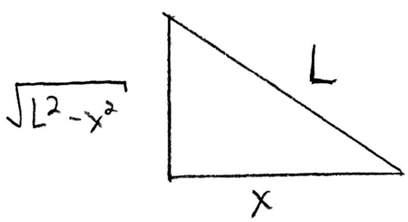
$$= \boxed{-\frac{1}{4}}$$

6. [20 pts] A man at point O on a pier is holding a rope of length L that is attached to a boat L units to his right. He begins walking up along the pier and he pulls the boat by a rope of length L, keeping the rope straight and taut as he pulls. The path followed by the boat has the property that the rope is always tangent to the curve[1]. Let $f(x)$ denote y position of the rowboat when it is a horizontal of x from the pier.



I. [3 pts] Let $m_{tan}(x)$ denote the slope of the tangent line at (x, y). Using the image above, argue why:

$$m_{tan} = -\frac{\sqrt{L^2 - x^2}}{x}.$$



The slope can be found using the shown triangle, from which it is clear

$$m = \frac{\text{rise}}{\text{run}} = -\frac{\sqrt{L^2-x^2}}{x}$$

II. [2 pts] Explain why the equation for the y-coordinate of the boat may be obtained from the initial value problem:

$$\frac{dy}{dx} = -\frac{\sqrt{L^2 - x^2}}{x}, \qquad y(L) = 0$$

The slope is also $\frac{dy}{dx}$, so $\frac{dy}{dx} = -\frac{\sqrt{L^2-x^2}}{x}$ is the relationship to be satisfied for all y on the curve. From the picture, note at $x = L$, $y = 0$, so the IC is $y(L) = 0$.

[1]We are implicitly assuming that the rowboat does not fishtail as it moves; it stays in line with the rope!

III. [12 pts] Find the specific solution to the initial value problem

$$\frac{dy}{dx} = -\frac{\sqrt{L^2 - x^2}}{x}, \qquad y(L) = 0.$$

$$dy = -\frac{\sqrt{L^2-x^2}}{x}\,dx$$

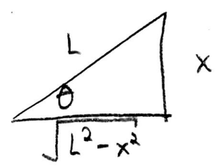
$$\int dy = -\int \frac{\sqrt{L^2-x^2}}{x}\,dx$$

The integral on the RHS requires a trig sub!

L^2-x^2 is of the form a^2-u^2 w/ $a=L$, $u=x$

a^2-u^2 suggests to use $u = a\sin\theta$. $\Rightarrow$ $x = L\sin\theta$

$dx = L\cos\theta\, d\theta$



$x = L\sin\theta$

$\sin\theta = \frac{x}{L}$

So: $y = \int -\frac{\sqrt{L^2-L^2\sin^2\theta}}{\cancel{L}\sin\theta}\cdot \cancel{L}\cos\theta\, d\theta$

$= \int -\frac{\sqrt{L^2\cos^2\theta}}{\sin\theta}\cos\theta\, d\theta$

$= \int -\frac{L\cos\theta}{\sin\theta}\cos\theta\, d\theta$

$= \int -\frac{L\cos^2\theta}{\sin\theta}\, d\theta$

$= \int -L\cdot\frac{1-\sin^2\theta}{\sin\theta}\, d\theta$

$= \int\left[-L\csc\theta + L\sin\theta\right]d\theta$

$= L\ln|\csc\theta + \cot\theta| - L\cos\theta + C$

$Y = L\ln\left|\frac{L}{x} + \frac{\sqrt{L^2-x^2}}{x}\right| - \cancel{L}\frac{\sqrt{L^2-x^2}}{\cancel{L}} + C$

$\underline{Y(L)=0:}$ $\quad 0 = L\,\cancel{\ln}|1|^{\,0} + 0 + C$.

$\Rightarrow C = 0$

So: $\boxed{y = L\ln\left|\frac{L}{x} + \frac{\sqrt{L^2-x^2}}{x}\right| - \sqrt{L^2-x^2}}$

IV. [3 pts] Set $L = 5$. Find $y'(3)$.

This is found immediately using the original ODE:

$$\frac{dy}{dx} = -\frac{\sqrt{L^2-x^2}}{x}$$

So when $L=5$, $x=3$:

$$\frac{dy}{dx} = -\frac{\sqrt{5^2-3^2}}{3}$$

$$= -\frac{\sqrt{16}}{3} = \boxed{-\frac{4}{3}}$$

Answers:

1. **Multiple choice**

   I. C.

   II. C.

   III. B.

2. **Multiselect** The choices A, B, E, F, G, H. should be circled!

3. I. $p_2(x) = 2(x+1) + \frac{3}{2}(x+1)^2$

   II. $a = -3, -1$

   III. Neither is a solution to the IVP: see the solutions for an explanation.

4. I. A. $xy^2 + 4y^2 = 1$

   B. $y' = -\dfrac{y^2}{4+2xy}$

   C. $y = \frac{1}{2}x + \frac{5}{2}$

   II. Converges

   A. $y' = \dfrac{2t^6}{8t^2-4}$

   B. $t = 1$

   C. $y = \frac{1}{2}x + \frac{5}{2}$

5. I. $f(x) = 6 + x + 6x^2 + \frac{3}{2}x^6 + \dots$

   II. A. $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$

   B. $-\frac{1}{4}$

6. I, II. See solutions

   III. $y = L \ln\left|\dfrac{L}{x} + \dfrac{\sqrt{L^2-x^2}}{x}\right| - \sqrt{L^2-x^2}$

   IV. $-4/3$