Math 1172

Sample Midterm 3

Spring 2016

Form A

Name: Solutions

OSU Username (name.nn): ____________

Lecturer: ____________

Recitation Instructor: ____________

Recitation Time: ____________

Instructions

- You have **55 minutes** to complete this exam. It consists of 6 problems on 8 pages including this cover sheet.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	12	
2	18	
3	15	
4	20	
5	15	
6	10	
Total	100	

1. **Multiple Choice** [12 pts]

   Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

   I. The polar form of a curve in the xy plane is given by $r = \cos(2\theta)$.

   Which of the following is the Cartesian description of the curve?

   A. $x^2 + y^2 = x$

   B. $x^2 + y^2 = 2x$

   C. $\boxed{(x^2 + y^2)^3 = (x^2 - y^2)^2}$

   D. None of the above

   $r = \cos(2\theta) = \cos^2\theta - \sin^2\theta$

   Mult by r^2 on both sides to get $r\cos\theta$, $r\sin\theta$ factors:

   $r^3 = r^2\cos^2\theta - r^2\sin^2\theta = (r\cos\theta)^2 - (r\sin\theta)^2$

   $(x^3+y^3)^{3/2} = x^2-y^2$

   $(x^3+y^3)^3 = (x^2-y^2)^2$

   II. A curve is described parametrically by:

   $$\begin{cases} x(t) &= 2\cos t \\ y(t) &= 3\sin t \end{cases}, \quad -\infty \le t \le \infty$$

   How many *distinct* vertical tangent lines does the curve have?

   A. 0

   B. 1

   C. $\boxed{2}$

   D. More than 2

   $\left.\begin{aligned} dx/dt &= -2\sin t \\ dy/dt &= 3\cos t \end{aligned}\right\} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos t}{-2\sin t}$

   We have vert asymp when $-2\sin t = 0$ but $3\cos t$ does not $\Rightarrow \sin t = 0 \to t = n\pi$ for integers n. However, no matter what n is, the vert. asymp are

   $x = \cos n\pi \Rightarrow x = 1$ or $x = -1$!

   III. Suppose that $\vec{u}$ is a nonzero vector, and $\hat{u}$ is a unit vector in the direction of $\vec{u}$. Then,

   A. $proj_{\vec{u}}\,\vec{u} = 0$.

   B. $\boxed{proj_{\vec{u}}\,\vec{u} = \vec{u}}$

   C. $proj_{\vec{u}}\,\vec{u} = \hat{u}$

   D. It depends on what the vector $\vec{u}$ is.

This should make sense intuitively if you understand what orthogonal projections are! It is also captured in the formula:

$$proj_{\vec{v}}\,\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

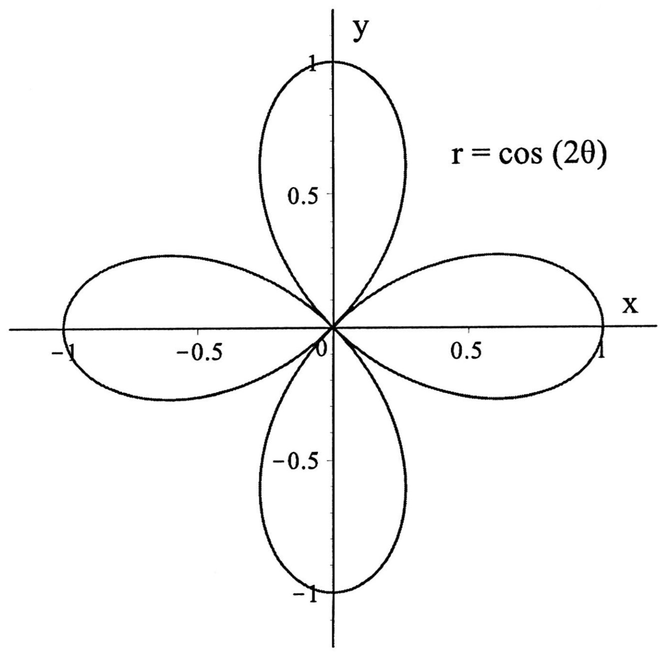
So: $proj_{\vec{u}}\,\vec{u} = \left(\frac{\vec{u}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right)\vec{u} = 1\,\vec{u}$!

2. **Multiselect** [18 pts]

Directions: Each problem below is worth 9 points. Circle *all* of the responses that MUST be true for each problem below. Note that there may be more than one correct response or even no correct responses!

A perfect answer for each part is worth 9 points. If you circle an incorrect choice, you will be penalized 3 points. If you do not circle a correct choice, you will be penalized 3 points. However, you cannot score below a 0 for either part of this problem. Thus, the possible scores for each part are 0, 3, 6, or 9 points.

I. The polar form of a curve in the xy plane is given by $r = \cos^2(\theta)$, which is shown below:



CIRCLE *all* of the following statements that MUST be true.

for all points where

A. $\dfrac{dy}{dx} \geq 0$ for all θ in $[0, 2\pi]$.

B. $\dfrac{dy}{dx} \geq 0$ [redacted] $x = \dfrac{1}{4}$.

C. $(x, y) = (0, 0)$ is on the curve

D. There is an angle θ where $\dfrac{dy}{dx} = 6$.

E. There are 4 points on the curve where there are horizontal tangent lines.

F. There are 4 points on the curve where there are vertical tangent lines.

There are 6!

II. [9 pts] Suppose that $\vec{u}$ and $\vec{v}$ are nonzero three dimensional vectors.

CIRCLE *all* of the following statements that *MUST* be true.

A. If $|\vec{u}| = |\vec{v}|$, then $\vec{u} = \vec{v}$.

B. $proj_{\vec{u}}\vec{v} = proj_{\vec{v}}\vec{u}$.

C. If $\vec{u}$ and $\vec{v}$ are parallel, then $\vec{u}\cdot\vec{v} = 0$.

$\boxed{\text{D. } \vec{u}\cdot(\vec{u}\times\vec{v}) = 0}$

E. If $\vec{u}$ and $\vec{v}$ are orthogonal, then $\vec{u}\times\vec{v} = 0$.

$\boxed{\text{F. } \vec{v}\cdot(\vec{u} - proj_{\vec{v}}\vec{u}) = 0}$

A: FALSE; A counterexample is $\vec{u} = \hat{i}$, $\vec{v} = \hat{j}$.

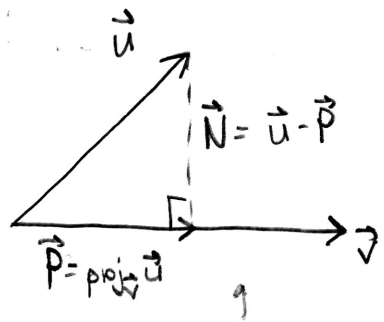
B FALSE; this is only the case if $\vec{u} \perp \vec{v}$ or $\vec{u} \parallel \vec{v}$. (in the former, both are $\vec{0}$), since $proj_{\vec{u}}\vec{v}$ must be in the direction of $\vec{u}$, $proj_{\vec{v}}\vec{u}$ must be in the direction of $\vec{v}$.

C. FALSE; Take $\vec{u} = \hat{i}$, $\vec{v} = 2\hat{i}$. It is true that if $\vec{u}, \vec{v}$ are parallel, then $\vec{u}\times\vec{v} = 0$. Since $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$ and $\theta = 0$ if $\vec{u}, \vec{v}$ are parallel.

D. TRUE; $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}, \vec{v}$, so $(\vec{u}\times\vec{v})\cdot\vec{u} = 0$.

E. FALSE; Take $\vec{u} = \hat{i}$, $\vec{v} = \hat{j}$. It is true that if $\vec{u}, \vec{v}$ are orthogonal, then $\vec{u}\cdot\vec{v} = 0$

F. TRUE:



Visual interpretation

-OR-

algebraically:

$$\vec{v}\cdot(\vec{u} - proj_{\vec{v}}\vec{u})$$

$$= \vec{v}\cdot\left(\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\right)$$

$$= \vec{v}\cdot u - \vec{v}\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\cdot\vec{v}$$

$$= \vec{u}\cdot\vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{\cancel{\vec{v}\cdot\vec{v}}}\right)\cancel{(\vec{v}\cdot\vec{v})}$$

$$= 0 \checkmark$$

3. [15 pts] Suppose that we define a new set of coordinates (u, v) by requiring that the usual Cartesian coordinates (x, y) are related to (u, v) as follows:

$$\begin{cases} x &= uv \\ y &= u^2 + v^2 \end{cases}$$

Now, suppose that a curve C is defined using these coordinates by the equation $u = v^2$.

I. Find the Cartesian coordinates (x, y) of the point on the curve when $v = 2$.

Since $u=v^2$, $x = uv = v^3$

$y = u^2+v^2 = v^4 + v^2$

So $x(2) = 2^3 = 8$

$y(2) = 2^4+2^2 = 20$ $\rightarrow$ $\boxed{(x,y) = (8,20)}$

II. Find $\dfrac{dy}{dx}$ when $v = 2$.

$\dfrac{dy}{dv} = 4v^3 + 2v$ so $\left.\dfrac{dy}{dv}\right|_{v=2} = 4(2)^3 + 2(2) = \underline{36}$

$\dfrac{dx}{dv} = 3v^2$ so $\left.\dfrac{dx}{dv}\right|_{v=2} = 3(2)^2 = \underline{12}$

So, $\left.\dfrac{dy}{dx}\right|_{v=2} = \left.\dfrac{dy/dv}{dx/dv}\right|_{v=2} = \dfrac{36}{12} = \boxed{3}$.

III. Find the Cartesian description of the tangent line to the curve when $v = 2$.

Express your final answer in the form $y = mx + b$.

$y - y(2) = m_{tan}(x - x(2))$

$y - 20 = 3(x - 8)$

$\boxed{y = 3x - 4}$

4. [20 pts] Suppose $\vec{r}(t) = \langle 2t^3, t^3 - 2, 1 - 2t^3 \rangle, \quad 0 \le t \le a$.

I. Find the unit tangent vector $\hat{T}(t)$ for this curve.

$$\vec{r}\,'(t) = \langle 6t^2, 3t^2, -6t^2 \rangle.$$

$$|\vec{r}\,'(t)| = \sqrt{(6t^2)^2 + (3t^2)^2 + (-6t^2)^2}$$

$$= \sqrt{81t^4}$$

$$= 9t^2$$

So, $\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\langle 6t^2, 3t^2, -6t^2 \rangle}{9t^2} \rightarrow \boxed{\hat{T}(t) = \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle}$

II. Find a value for a so the length of this curve is 24.

$$\text{length} = \int_0^a |\vec{r}\,'(t)|\, dt$$

$$24 = \int_0^a 9t^2\, dt$$

$$24 = 3t^3 \Big|_0^a$$

$$24 = 3a^3 - 0$$

$$8 = a^3$$

$$\boxed{a = 2}$$

III. Explain why the curve is not parameterized by arclength. Then, find another description of the curve that uses arclength as a parameter.

The curve is not parameterized by arclength since $|\vec{r}\,'(t)| \neq 1$ for all t.

To reparameterize:

Step 1: Find $|\vec{r}\,'(t)|$.

Done already; $|\vec{r}\,'(t)| = 9t^2$

Step 2: Find $s(t)$ from $s(t) = \int_0^t |\vec{r}\,'(\tau)|\, d\tau$

$$s = \int_0^t 9\tau^2 \, d\tau$$

$$s = 3t^3$$

Step 3: Find t as a function of s

$$s = 3t^3 \rightarrow t = \left(\frac{s}{3}\right)^{1/3}$$

← Here, you could solve for t^3 since $\vec{r}(t)$ only has t^3 terms!

Step 4: Substitute into $\vec{r}(t)$.

$$\vec{r}(t) = \langle 2t^3, t^3 - 2, 1 - 2t^3 \rangle \rightarrow \vec{r}(s) = \left\langle 2\left[\left(\frac{s}{3}\right)^{1/3}\right]^3, \left[\left(\frac{s}{3}\right)^{1/3}\right]^3 - 2, 1 - 2\left[\left(\frac{s}{3}\right)^{1/3}\right]^3 \right\rangle$$

$$\boxed{\vec{r}(s) = \left\langle \frac{2s}{3}, \frac{1}{3}s - 2, 1 - \frac{2s}{3} \right\rangle}$$

IV. Find the parametric equations of the tangent vector to this curve at $t = 1$.

When $t = 1$: $\vec{r}(1) = \langle 2(1)^3, (1)^3 - 2, 1 - 2(1)^3 \rangle$

$$= \langle 2, -1, -1 \rangle$$

From I $\quad \vec{r}\,'(1) = \langle 6(1)^2, 3(1)^2, -6(1)^2 \rangle = \langle 6, 3, -6 \rangle$.

So, the tan line $\vec{R}(t)$ is given by:

$$\vec{R}(t) = [\vec{r}\,'(1)]t + \vec{r}(1)$$

$$\vec{R}(t) = \langle 6, 3, -6 \rangle t + \langle 2, -1, -1 \rangle$$

$$= \boxed{\langle 6t + 2, 3t - 1, -6t - 1 \rangle}$$

5. [10 pts] Suppose $f(x,y) = \dfrac{2x^4 - x^2y^2}{x^2y^2 + 2y^4}$.

Determine whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists or does not exist. Explain your answer!

Along $x=0$, $f(0,y) = \dfrac{0}{2y^4} = 0$.

Along $y=2x$: $f(x,2x) = \dfrac{2x^4 - x^2(2x)^2}{x^2(2x)^2 + 2(2x)^4} = \dfrac{2x^4 - 4x^4}{4x^4 + 32x^4} = \dfrac{-2\cancel{x^4}}{36\cancel{x^4}} = -\dfrac{1}{18}$.

The function tends to different values along different paths as $(x,y)\to(0,0)$ so the limit DNE!

5. [10 pts] Suppose $f(x,y) = \dfrac{2x^4 - x^2y^2}{x^2y^2 + 2y^4}$.

Determine whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists or does not exist. Explain your answer!

Try the Two Path Test with lines of the form $y = mx$
(since the degrees of numerator, denom are the same!).

$\underline{x=0}$: $f(0,y) = \dfrac{0}{2y^4} = 0$ for all $y \neq 0$.

So, along $x=0$, $f(x,y) \to 0$ as $(x,y) \to (0,0)$

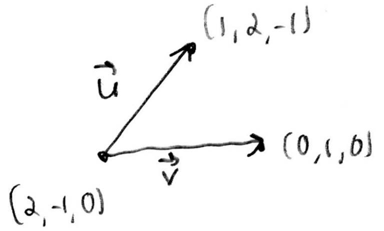
$\underline{y=x}$: $f(x,x) = \dfrac{2x^4 - x^2(x^2)}{x^2(x^2) + 2x^4} = \dfrac{x^4}{3x^4} = \dfrac{1}{3}$ for all $x \neq 0$.

So, along $y=x$, $f(x,y) \to \frac{1}{3}$ as $(x,y) \to (0,0)$

Since $f(x,y)$ approaches different values as $(x,y) \to (0,0)$ along different paths, the limit DNE.

6. [15 pts]

I. Find the equation of the plane that passes through the points $(2,-1,0)$, $(1,2,-1)$, and $(0,1,0)$.



$\vec{u} = \langle 1-2, 2-(-1), -1-0 \rangle = \langle -1, 3, -1 \rangle$

$\vec{v} = \langle 0,1,0 \rangle - \langle 2,-1,0 \rangle = \langle -2, 2, 0 \rangle.$

$\vec{u} \times \vec{v}$ is a vector $\perp$ the plane:

$$\vec{u}\times\vec{v} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & -1 \\ -2 & 2 & 0 \end{bmatrix} = 2\hat{i} + 2\hat{j} + 4\hat{k}$$

The equation of the plane is:

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0.$$

Here, $a = 2$, $b = 2$, $c = 4$.

For (x_0, y_0, z_0), use any point on the plane; say $(0,1,0)$:

$$2(x-0) + 2(y-1) + 4(z-0) = 0$$

$$\boxed{x + y + 2z = 1}$$

II. Show that the curve $\vec{r}(t) = \langle t^2+3, 2t^3-t^2-2, -t^3 \rangle$ lies on the plane $x+y+2z=1$.

If a point lies on the curve: $x = t^2+3$, $y = 2t^3 - t^2 - 2$, $z = -t^3$.

If a point lies on the plane: $x + y + 2z = 1$.

Thus, for the curve to lie on the plane; we check that for any t: if $x(t) + y(t) + 2z(t) = 1$:

$$x+y+2z = [t^2+3] + [2t^3 - t^2 - 2] + 2[-t^3]$$

$$= \cancel{t^2} + 3 + \cancel{2t^3} - \cancel{t^2} - 2 - \cancel{2t^3}$$

$$= 1 \checkmark$$

Thus, $\vec{r}(t)$ lies on the plane!