DISCLAIMER

Solutions

General Information:

This midterm is a *sample* midterm. This means:

- The sample midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to the actual midterm. Note that this may be a departure from the format used on exams in previous semesters!
- The sample midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take the sample exam:

The sample midterm should be treated like the actual exam. This means:

- "Practice like you play." Schedule 55 minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.
- The problems on this exam are mostly based on the Worksheets posted on the Math 1152 website and your previous quizzes.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and take-home quizzes with this intent, and there could be a problem on the exam that requires you to apply the material in an unfamiliar setting. To aid in preparation, there is such a problem on this sample exam.

How to use the solutions:

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

Math 1152 | Name: ______________________

Sample Midterm 3 | OSU Username (name.nn): ______________________

Autumn 2016 | Lecturer: ______________________

Recitation Instructor: ______________________

Form B | Recitation Time: ______________________

Instructions

- You have **55 minutes** to complete this exam. It consists of 6 problems on 10 pages including this cover sheet. Page 11 has possibly helpful formulas and may also be used for extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work!** Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	20	
2	12	
3	20	
4	30	
6	18	
Total	100	

1. **Multiple Choice** [20 pts]

   Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

   I. Which of the following functions is a general solution to the differential equation:

$$\frac{dy}{dt} + 2y = 2?$$

A. $y(t) = e^{-2t} + C$ | $\boxed{\text{B. } y(t) = Ce^{-2t} + 1}$ | C. $y(t) = t^2 + C$

D. $y(t) = C\cos(2t) + 2$ | E. $y(t) = C\sin(2t) + 2$ | F. None of the above

· If $y(t) = Ce^{-2t} + 2$
$y'(t) = -2Ce^{-2t}$

so $y' + 2y = -2Ce^{-2t} + 2[Ce^{-2t} + 1]$
$= 2$ ✓.

   II. Let $f(x)$ be a continuously differentiable function whose power series is:

$$f(x) = \sum_{k=1}^{\infty} a_k(x-4)^k.$$

Let R be the radius of convergence for this series, and suppose it is known that the series $\sum_{k=1}^{\infty} a_k$ converges. Then it MUST be true that:

A. $R \leq 1$ | B. $R = 1$ | $\boxed{\text{C. } R \geq 1}$

D. $R = \infty$; i.e. the series converges for all x.

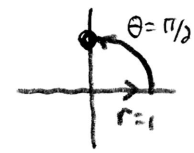
E. None of these.

Note: $f(5) = \sum_{k=1}^{\infty} a_k(5-4)^k = \sum_{k=1}^{\infty} a_k(1)^k = \sum_{k=1}^{\infty} a_k$

Since $x=5$ is 1 unit away from the center of the series $(x=4)$, the radius of convergence must be at least 1.

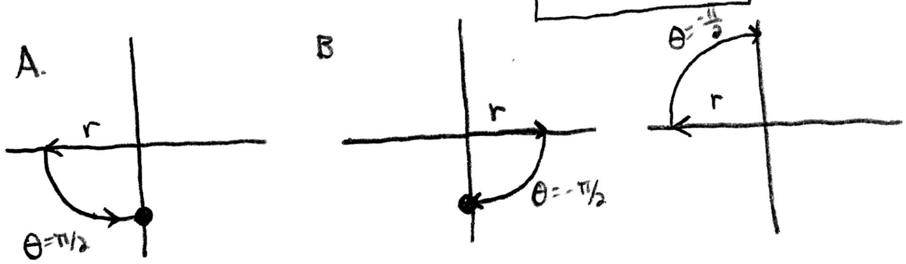
(It could be greater; for instance, if $a_k = \frac{1}{k!}$, $\sum a_k$ converges by Ratio test, and $\sum \frac{1}{k!}(x-4)^k$ can be shown to have $R=\infty$.
If $a_k = \frac{(-1)^k}{k}$, it can be shown $\sum a_k$ converges, but the ROC is exactly 1).

III. A point in the xy-plane is described by the *polar coordinates* $(r, \theta) = \left(1, \frac{\pi}{2}\right)$. →



Which of the following gives an alternate description of the point in polar coordinates?

A. $\left(-1, \frac{\pi}{2}\right)$ B. $\left(1, -\frac{\pi}{2}\right)$ $\boxed{\text{C. } \left(-1, -\frac{\pi}{2}\right)}$ D. None of the above



IV. What is the radius of convergence for the Taylor series centered at $x = 0$ for $f(x) = \dfrac{1}{1-4x}$?

A. 0 $\boxed{\text{B. } 1/4}$ C. 1 D. 4 E. None of the above.

The series for $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ converges for $|x|<1$ so the series for $\frac{1}{1-4x}$ converges when $|4x|<1 \Rightarrow \boxed{|x|<\frac{1}{4}}$

V. Which of the following is the Taylor series centered at $x = 0$ for $f(x) = x\cos(x)$?

A. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$ B. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ C. $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+1)!}$

D. $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k)!}$ E. $\displaystyle\sum_{k=0}^{\infty} x^{2k+1}$ $\boxed{\text{F. None of the above.}}$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$x\cos x = x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k+1}.$$

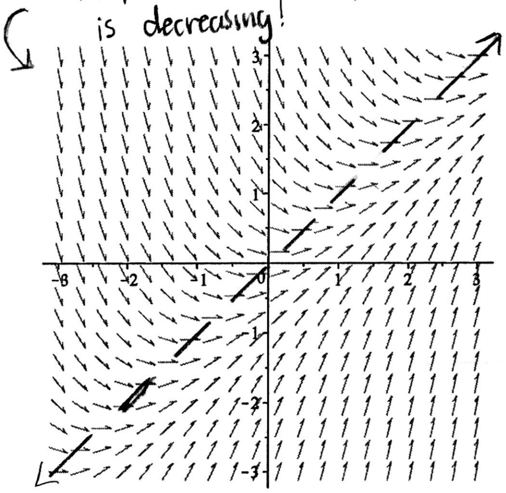
* KNOW the series for e^x, $\cos x$, $\sin x$, $\frac{1}{1-x}$ both in summation notation and be able to write out several terms!

2. [12 pts] (Short Answer)

I. Explain whether the following image could be the direction field field for the differential equation:

$$\frac{dy}{dx} = y - x.$$

Any solution above $y=x$ is decreasing!



$y=x$.

This cannot be a direction field; Note when $y-x>0$, we have $y>x$. Thus, $\frac{dy}{dx} = y-x > 0$ for any points above the line $y=x$, which means that any solution above $y=x$ should always be increasing!

II. The curve C is defined via the polar equation $r = 4\sec\theta$. Write a description of this curve in the Cartesian coordinates x, and y.

$r = 4 \sec\theta$

$r = 4 \frac{1}{\cos\theta}$

$\underbrace{r\cos\theta}_{x} = 4 \Rightarrow \boxed{x=4}$

III. Suppose that $f(x)$ is an infinitely differentiable function and it is known that $f(x)$ has a relative maximum at $x = 3$. Explain why the Taylor Series for $f(x)$ centered at $x = 3$ must be of the form:

$$a_0 + a_2(x-3)^2 + a_3(x-3)^3 + \ldots$$

i.e. why the coefficient $a_1 = 0$. What must be the value of the relative maximum?

Since $f(x)$ is differentiable, $f'(x) = 0$ at any relative extrema. Thus, $f'(3) = 0$. But, $a_1 = \frac{f'(3)}{1!}$ by definition, so $\boxed{a_1 = 0}$

At $x=3$, $f(3) = a_0 + a_2 \cancel{(3-3)^2}^{0} + a_3 \cancel{(3-3)^3}^{0} + \ldots$

$\boxed{f(3) = a_0}$

3. [20 pts] A curve is described parametrically by:

$$\begin{cases} x(t) &= t^3 - 3t \\ y(t) &= 1 - \dfrac{12}{t} \end{cases}$$

for all t where $x(t)$ and $y(t)$ are well-defined.

I. Find $\dfrac{dy}{dx}$ in terms of t. Simplify your final answer!

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$x = t^3 - 3t$ $\qquad$ $y = 1 - \frac{12}{t}$

$\frac{dx}{dt} = 3t^2 - 3$ $\qquad$ $\frac{dy}{dt} = \frac{12}{t^2}$

So: $\dfrac{dy}{dx} = \dfrac{12/t^2}{3t^2-3} = \dfrac{12}{3t^2(t^2-1)} = \boxed{\dfrac{4}{t^2(t^2-1)}}$

II. Give the Cartesian equation(s) for any vertical tangent lines to the curve or state that there are none.

Vertical Tan Lines
- Slope is infinite
- Eqn: $x = $ const
- The function and tan line must agree at pt of tangency

$\frac{dy}{dx}$ is infinite when $t^2(t^2-1) = 0$
$t^2(t+1)(t-1) = 0 \Rightarrow t = -1, 0, 1$

Note when $t = 0$, $y(t)$ is undefined; there is a vertical asymptote (not a vertical tangent line) there since $\lim_{t \to 0^+} y(t) = -\infty$!

The curve is defined when $t = \pm 1$; so there are vertical tangent lines there.

$t = 1$: $x = x(1) = (1)^3 - 3(1) \to \boxed{x = -2}$

$t = -1$: $x = x(-1) = (-1)^3 - 3(-1) \to \boxed{x = 2}$

III. Give the Cartesian equation(s) for any horizontal tangent lines to the curve or state that there are none.

$\frac{dy}{dx} = 0$ when the numerator is 0. But $\frac{dy}{dx} = \dfrac{4}{t^2(t^2-1)}$

so $\frac{dy}{dx} \neq 0$ for any t!

$\to$ $\boxed{\text{No horizontal tangent lines}}$

IV. Find the Cartesian equation for the tangent line to the curve when $t = 2$.

When $t=2$, $\quad \dfrac{dy}{dx} = \dfrac{4}{(2)^2[(2)^2-1]} = \dfrac{1}{3}.$

$$x(2) = (2)^3 - 3(2) = 2$$

$$y(2) = 1 - \frac{12}{(2)} = -5.$$

The tangent line is:

$$y - y(2) = m\left[x - x(2)\right]$$

$$y - (-5) = \frac{1}{3}\left[x - 2\right].$$

$$\boxed{y = \frac{1}{3}x - \frac{17}{3}}$$

V. Find the Cartesian equations of any horizontal asymptotes to the curve or state that there are none.

Horizontal asymptotes are found by computing $\lim\limits_{x \to \pm\infty} y(x)$.

Note that $\lim\limits_{t\to\infty} x(t) = \lim\limits_{t\to\infty} (t^3 - 3t) = +\infty$

$\lim\limits_{t\to-\infty} x(t) = \lim\limits_{t\to-\infty} (t^3 - 3t) = -\infty.$

Thus, we can find horizontal asymptotes by computing

$\lim\limits_{t\to\infty} y(t)$, $\lim\limits_{t\to-\infty} y(t)$.

In either case:

$$\lim_{t\to\pm\infty} y(t) = \lim_{t\to\pm\infty}\left(1 - \frac{12}{t}\right) = 1.$$

Thus, $\boxed{\text{the horizontal asymptotes are } y = 1}$ as $x \to \infty$ $(t \to \infty)$
$x \to -\infty$ $(t \to -\infty)$

4. [30 pts] (Taylor Series)

I. Given the power series for a function $f(x)$ is given by:

$$f(x)=\sum_{k=0}^{\infty}\frac{(x+2)^{2k}}{(2k+1)\,4^k}$$

series is in powers of $(x+2)$. Since "centered at $x=c$" means to express in powers of $(x-c)$, this means $c=-2$.

A. State the center of the series.

The series is centered at $x=-2$.

B. Find $f(-2)$.

Write out a few terms

$$f(x) = 1+\frac{1}{12}(x+2)^2+\ldots$$

$$\boxed{f(-2)=1}$$

C. Find the radius of convergence for the power series for $f(x)$.

Use ratio test to check for absolute conv.

$$L(x)=\lim_{k\to\infty}\left|\frac{(x+2)^{2(k+1)}}{[2(k+1)+1]\,4^{k+1}}\cdot\frac{4^k(2k+1)}{(x+2)^{2k}}\right|$$

$$=\lim_{k\to\infty}\left|\frac{(x+2)^{2k+2}}{(x+2)^{2k}}\cdot\frac{4^k}{4^{k+1}}\cdot\frac{2k+1}{2k+3}\right|$$

$$=\lim_{k\to\infty}\left|\frac{\cancel{(x+2)^{2k}}(x+2)^2}{\cancel{(x+2)^{2k}}}\cdot\frac{\cancel{4^k}}{\cancel{4^k}4}\cdot\frac{2k+1}{2k+3}\right|$$

$$=\frac{1}{4}|x+2|^2\lim_{k\to\infty}\frac{2k+1}{2k+3}$$

$$L(x)=\frac{1}{4}|x+2|^2(1)$$

WARNING: Absolute values are NOT linear:

$|x+2|<2 \not\Rightarrow |x|<0$!

This will be penalized harshly!

Ratio test ensures the series converges when $L(x)<1$, diverges if $L(x)>1$ so the ROC is found by setting $\frac{1}{4}|x+2|^2<1$

$$|x+2|^2<4$$

$$|x+2|<2$$

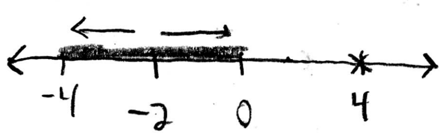
$$\boxed{\text{The ROC is } 2}$$

D. Let $g(x) = f'(x)$. What is the radius of convergence for the Taylor series for $g(x)$? Justify your response!

The ROC for $f'(x)$ is 2; differentiating does not change the ROC!

E. Does the series for $g(4)$ converge or diverge? Explain your answer!

The series for $g(x)$ is centered at $x=-2$ and has ROC 2 by D. The series will thus converge for any x in $(-4,0)$, diverge for $x<-4$ or $x>0$ (The endpts $x=0$, $x=-4$ would need to be checked separately, but are not relevant here!).



$x=4$ is outside the open interval of convergence!

Hence, the series for $g(4)$ diverges

F. Find $g'(-2)$.

$$f(x) = 1 + \frac{1}{12}(x+2)^2 + \frac{1}{80}(x+2)^4 + \dots$$

$$g(x) = f'(x) = \frac{1}{6}(x+2) + \frac{1}{20}(x+2)^3 + \dots$$

$$g'(x) = \frac{1}{6} + \frac{3}{20}(x+2)^2 + \dots$$

$$\rightarrow \boxed{g'(-2) = \frac{1}{6}}$$

G. Find $g^{(19)}(-2)$.

$g^{(19)}(-2) = f^{(20)}(-2)$ since $g(x) = f'(x)$ ← 19 deriv of $g(x)$ are thus 20 deriv of $f(x)$.

We know $a_{20} = \dfrac{f^{(20)}(-2)}{20!}$ so $f^{(20)}(-2) = 20!\, a_{20}$.

Also a_{20} is the coeff of $(x+2)^{20}$. since $(x+2)^{2k} = (x+2)^{20}$ when $k=10$,

$$a_{20} = \left.\frac{1}{(2k+1)4^k}\right|_{k=10} = \frac{1}{21 \cdot 4^{10}}$$

So: $$\boxed{f^{(20)}(-2) = 20! \cdot \frac{1}{21 \cdot 4^{10}}}$$

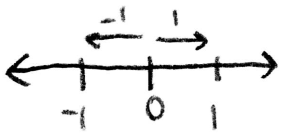
II. Given that the Taylor series centered at $x = 0$ for $\ln(1 + x)$ is given by:

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$$

and has radius of convergence 1:

A. Find the interval of convergence for the Taylor series for $\ln(1 + x)$.

The series is centered at $x=0$ and has ROC 1, so the series converges when $-1 < x < 1$.



We must check the endpts separately:

$x=-1$: $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{2k}(-1)}{k} = \sum_{k=1}^{\infty} \frac{-1}{k}$.

This is the harmonic series so it diverges!

$x=1$: $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ ← This is alternating and: $\lim_{k\to\infty} \frac{1}{k} = 0$, $\frac{1}{k}$ is decreasing

so this converges by alt series test

Series div when $x=-1$
conv when $x=1$

$\Rightarrow$ $\boxed{\text{IOC is } -1 < x \leq 1 \text{ or } (-1, 1]}$

B. Write out the Taylor series centered at $x = 0$ in summation notation for the function:

$$f(x) = \int_0^x t^2 \ln(1 + t)\, dt.$$

What is the interval of convergence for this series?

$$t^2 \ln(1+t) = \underbrace{t^2 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{t^k}{k}} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{t^{k+2}}{k}.$$

If you try to integrate this product, good luck! Do algebra first!

So, $$\int_0^x t^2 \ln(1+t)\,dt = \int_0^x \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} t^{k+2}\, dt$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \int_0^x t^{k+2}\, dt$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+3)} t^{k+3} \Big|_0^x$$

$$= \boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2+3k} x^{k+3}}.$$

[OVER] ↗

The ROC will not change when integrating, so we know the series will still converge for $-1 < x < 1$.

Check endpoints:

$x=1$: $f(1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2+3k} (1)^{k+3} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2+3k}$.

This is alternating and $\frac{1}{k^2+3k}$ is decreasing and $\lim_{k\to\infty} \frac{1}{k^2+3k} = 0$

so the series for $f(1)$ converges.

$x=-1$: $f(-1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2+3k} (-1)^{k+3} = \sum_{k=1}^{\infty} \frac{(-1)^{2k+4}}{k^2+3k}$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{2k}(-1)^4}{k^2+3k}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^2+3k}.$$

Since $\frac{1}{k^2+3k} > 0$, use LCT with $\sum_{k=1}^{\infty} \frac{1}{k^2}$:

Note: $\lim_{k\to\infty} \frac{1/k^2}{1/(k^2+3k)} = \lim_{k\to\infty} \frac{k^2+3k}{k^2} = 1$

so both $\sum \frac{1}{k^2}$ and $\sum \frac{1}{k^2+3k}$ will converge or both will diverge by LCT.

Since $\sum \frac{1}{k^2}$ is a p-series w/ $p=2>1$, it converges, so $\sum \frac{1}{k^2+3k}$ converges as well.

$$\boxed{\text{The IOC is thus } -1 \le x \le 1 \text{ or } [-1, 1]}$$

* Integrating / Differentiating **NEVER** changes the ROC but can change IOC.

5. [22 pts] Consider the differential equation $\dfrac{dy}{dt} = \dfrac{3\cos^3 t}{2y}$.

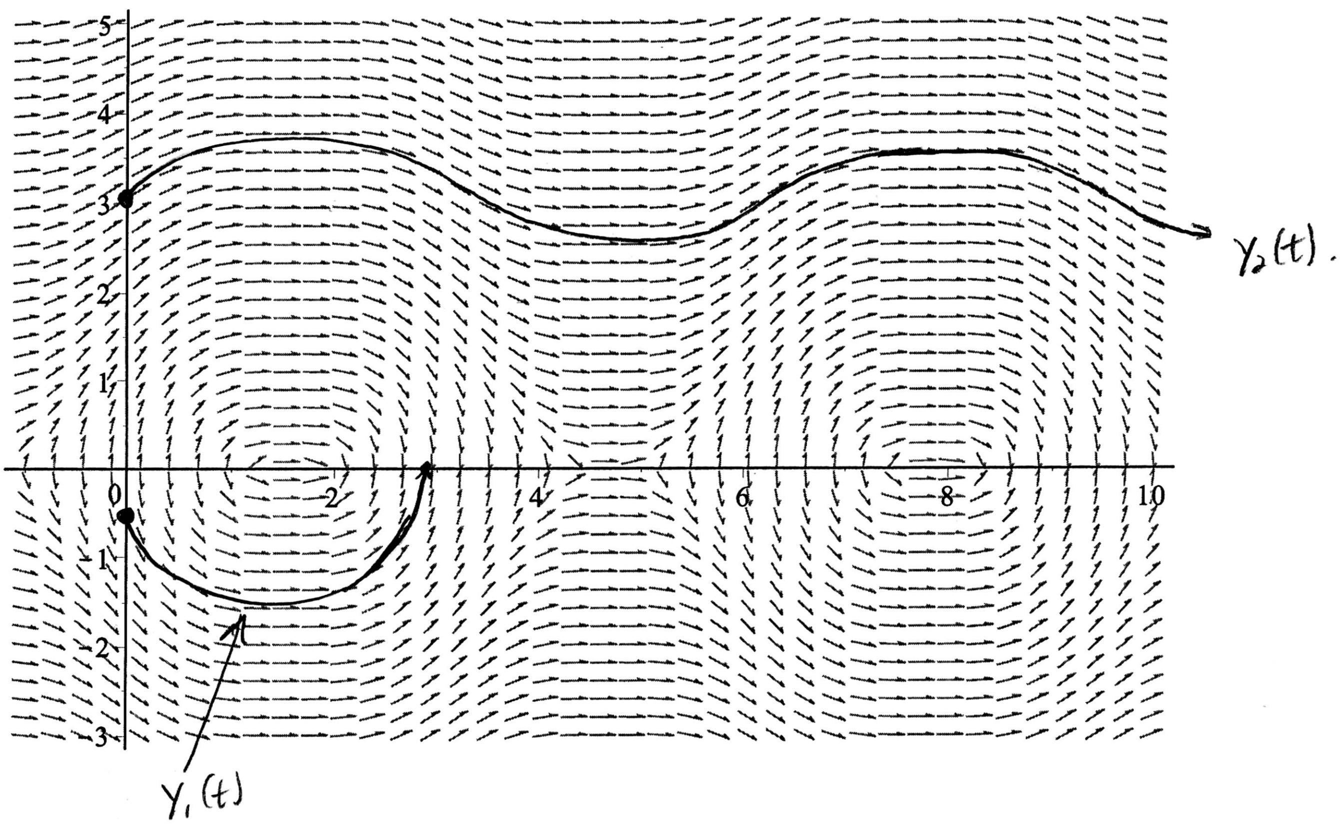
I. [1 pts] Give the order of the equation and state whether it is linear or nonlinear.

First order, non-linear.

highest deriv is $\frac{dy}{dt}$ — $2yy' = 3\cos^3 t$ is not linear in y, y'!

II. [3 pts] Let $y_1(t)$ be the solution to the differential equation with $y(0) = -\frac{1}{2}$ and $y_2(t)$ be the solution to the differential equation with $y(0) = 3$.

A. On the direction field below, sketch the graphs of $y_1(t)$ and $y_2(t)$ for $t \geq 0$.



B. From your direction field, does there appear to be a maximum time for which the solutions are defined? Let t_1 be the largest time for which $y_1(t)$ is defined and t_2 be the largest time for which $y_2(t)$ is defined. Using the direction field, estimate t_1 and t_2 or state that the solutions appear to be defined for all t.

From the direction field: • $t_1 \approx 3$

• y_2 is defined for all t.

III. [9 pts] Find the **general solution** to the differential equation. To get full credit, you must solve for y explictly!

$2y\,dy = 3\cos^3 t\,dt$

use trig sub: $\cos^3 t = \cos^2 t \cos t = (1-\sin^2 t)\cos t$

$u = \sin t$

$du = \cos t\,dt$.

$\int 2y\,dy = 3\int \cos^3 t\,dt$

$y^2 = 3\int (1-\sin^2 t)\cos t\,dt$

$= 3\int (1-u^2)\,du$

$= 3u - u^3 + C$

$y^2 = 3\sin t - \sin^3 t + C$

$\boxed{y = \pm\sqrt{3\sin t - \sin^3 t + C}}$

The $\pm$ must be resolved using IC!

IV. [4 pts] Find the **specific solutions** $y_1(t)$ and $y_2(t)$ (defined on the previous page).

$y_1(0) = -\frac{1}{2}$

$\Rightarrow$ $\cancel{+}\frac{1}{2} = \cancel{+}\sqrt{3\sin 0 - \sin^3(0) + C}$

$\frac{1}{4} = C$

$\boxed{y_1(t) = -\sqrt{3\sin t - \sin^3 t + \frac{1}{4}}}$

$y_2(0) = 3$

$3 = +\sqrt{3\sin 0 - \sin^3(0) + C}$

$9 = C$

$\boxed{y_2(t) = +\sqrt{3\sin t - \sin^3 t + 9}}$

V. [5 pts] From your solution, argue why $t_1 < \frac{3\pi}{2}$ by showing that $y(t)$ is not defined when $t = \frac{3\pi}{2}$ and why $y_2(t)$ is defined for all t. Compare with the curves you drew in B.

Note that when $t = \frac{3\pi}{2}$, the expr under $\sqrt{\ }$ is $3\sin t - \sin^3 t + \frac{1}{4} = -3 + 1 + \frac{1}{4} < 0$!

So $y_1(t)$ is not defined when $t = \frac{3\pi}{2}$

$\Rightarrow t_1 < \frac{3\pi}{2}$ (since t_1 is the max time for which $y_1(t)$ is defined)!

We predicted $t_1 \approx 3 < \frac{3\pi}{2}$ ✓

Note that $3\sin t - \sin^3 t + 9$

$> -3 - 1 + 9$

> 5

So the expression under the radical is always positive so $y_2(t)$ is always defined!