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General Information:

This midterm is a *sample* midterm. This means:

- The sample midterm contains problems that are of similar, but not identical, difficulty to those that will be asked on the actual midterm.
- The format of this exam will be similar, but not identical to the actual midterm. Note that this may be a departure from the format used on exams in previous semesters!
- The sample midterm is of similar length to the actual exam.
- Note that there are concepts covered this semester that do *NOT* appear on this midterm. This does not mean that these concepts will not appear on the actual exam! Remember, this midterm is only a sample of what *could* be asked, not what *will* be asked!

How to take the sample exam:

The sample midterm should be treated like the actual exam. This means:

- "Practice like you play." Schedule 55 minutes to take the sample exam and write answers as you would on the real exam; include appropriate justification, calculation, and notation!
- Do not refer to your books, notes, worksheets, or any other resources.
- You should not need (and thus, should not use) a calculator or other technology to help answer any questions that follow.
- The problems on this exam are mostly based on the Worksheets posted on the Math 1172 website and your previous quizzes.

  However, in your future professions, you will need to use mathematics to solve many different types of problems. As such, part of the goal of this course is:

  - to develop your ability to understand the broader mathematical concepts (not to encourage you just to memorize formulas and procedures!)
  - to apply mathematical tools in unfamiliar situations (Indeed, tools are only useful if you know when to use them!)

  There have been questions in your online homework and take-home quizzes with this intent, and there could be a problem on the exam that requires you to apply the material in an unfamiliar setting. To aid in preparation, there is such a problem on this sample exam.

How to use the solutions:

- Work each of the problems on this exam *before* you look at the solutions!
- *After* you have worked the exam, check your work against the solutions. If you are miss a type of question on this midterm, practice other types of problems like it on the worksheets!
- If there is a step in the solutions that you cannot understand, please talk to your TA or lecturer!

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Math 1172

Sample Midterm 3

Spring 2016

Form B

Name: ____________________

OSU Username (name.nn): ____________________

Lecturer: ____________________

Recitation Instructor: ____________________

Recitation Time: ____________________

Instructions

- You have **55 minutes** to complete this exam. It consists of 6 problems on 9 pages including this cover sheet. Page 11 has possibly helpful formulas and may also be used for extra workspace.
- If you wish to have any work on the extra workspace pages considered for credit, indicate in the problem that there is additional work on the extra workspace pages and **clearly label** to which problem the work belongs on the extra pages.
- The value for each question is both listed below and indicated in each problem.
- Please **write clearly** and make sure to **justify your answers** and **show all work**! Correct answers with no supporting work may receive no credit.
- You may not use any books or notes during this exam
- Calculators are NOT permitted. In addition, neither PDAs, laptops, nor cell phones are permitted.
- Make sure to read each question carefully.
- Please **CIRCLE** your final answers in each problem.
- A random sample of graded exams will be copied prior to being returned.

Problem	Point Value	Score
1	12	
2	18	
3	25	
4	15	
5	15	
6	25	
Total	100	

1. **Multiple Choice** [12 pts]

Circle the response that best answers each question. Each question is worth 4 points. There is no penalty for guessing and no partial credit.

I. Consider the segment of the curve given by $\vec{r}(t) = \langle 2t, 4t^2 \rangle$, $0 \le t \le 1$.

Suppose that the vector-valued function $\vec{r}(\tau) = \langle 4\tau^2, 16\tau^4 \rangle$, $0 \le \tau \le a$ describes the same segment of the curve. Then,

A. $a = 1$ $a = 2$ $\boxed{\text{C. } a = \frac{\sqrt{2}}{2}}$ D. None of the above

By comparing $x(t), x(\tau)$: $2t = 4\tau^2 \rightarrow t = 2\tau^2$. So, t is increasing in τ, and thus $t=1$ will correspond to $\tau = a$; so

$$1 = 2a^2$$

$$a = \pm\sqrt{\tfrac{1}{2}} = \pm \frac{\sqrt{2}}{2}$$

Since $a > 0$, choose $\underline{a = +\frac{\sqrt{2}}{2}}$

II. Which of the following is the equation of a plane that passes through $(1, -2, -4)$ and is parallel to the plane $2x + y - 3z = 3$?

A. $2x + y - 3z = 3$ $\boxed{\text{B. } 2x + y - 3z = 12}$

C. $x - 2y - 4z = 12$ D. None of the above

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0.$$

- The plane is parallel to $2x + y - 3z = 3 \rightarrow \underline{\langle a, b, c \rangle = \langle 2, 1, -3 \rangle}$.
- The plane passes through $(1,-2,-4)$ so $\underline{(x_0, y_0, z_0) = (1, -2, -4)}$.

The plane is given by $2(x-1) + 1(y+2) - 3(z+4) = 0 \rightarrow \underline{2x + y - 3z = 12}$

III. Suppose that $\vec{r}(t) = \langle t^2, 4t^4 \rangle$, $-\infty \le t \le \infty$. This curve is:

A. The entire parabola $y = 4x^2$. $\boxed{\text{B. The right half of } y = 4x^2.}$

C. The left half of $y = 4x^2$. D. None of the above.

$$x(t) = t^2$$

$$y(t) = 4t^4 = 4(t^2)^2$$

$$\rightarrow \underline{y = 4x^2}.$$

Note $x(t) = t^2$, so $x(t) \ge 0$ for all t!

2. **Short Answer** [18 pts]

Directions: Answer the following questions. Each question is worth 3 points. You do not need to justify your answer. There is no partial credit and no penalty for guessing.

Consider the function $z = f(x, y) = \sqrt{16 - 2x^2 + y^2}$.

I. State the domain of $f(x, y)$. Need the expression under $\sqrt{}$ to be nonnegative:

$$\boxed{\{(x,y) \mid 16-2x^2+y^2 \geq 0\}}$$

II. State the range of $f(x, y)$.

The range of $\sqrt{u}$ is $[0,\infty)$. Note when $u=0$, $\sqrt{u}=0$ and there are x, y such that $16-2x^2+y^2=0$ (take $x=4, y=4$). Also, $\sqrt{u} \to \infty$ as $u \to \infty$ letting $x=0$ and $y \to \infty$, $\sqrt{16-2x^2+y^2} \to \infty$. The range is thus $\boxed{[0,\infty)}$

III. Give the xz-trace of the surface.

For xz-trace: set $y=0$. $\Rightarrow$ $z=\sqrt{16-2x^2+0}$

$$\boxed{z=\sqrt{16-2x^2}}$$

IV. Give the equation of the level curve $z = 4$.

$$Z=4=\sqrt{16-2x^2+y^2}$$

$$\cancel{16}=\cancel{16}-2x^2+y^2$$

$$\boxed{2x^2=y^2}$$

V. Determine $\displaystyle\lim_{(x,y)\to(2,1)} f(x, y)$.

$$\lim_{(x,y)\to(2,1)} \sqrt{16-2x^2+y^2} = \sqrt{16-2(2)^2+(1)^2} = \sqrt{16-8+1}$$

$$= \sqrt{9} = \boxed{3}$$

VI. Find a function $f(t)$ such that the curve $\vec{r}(t) = \langle 2t, f(t), t\rangle$ lies on the surface.

Suppose (x,y,z) is a point on the curve and the surface.

(x,y,z) is on the curve: so $x=2t$, $y=f(t)$, $z=t$ for some t.

(x,y,z) is on the surface: so $z=\sqrt{16-2x^2+y^2}$

Combining: $t=\sqrt{16-2(2t)^2+[f(t)]^2}$ $\to$ $t^2=16-8t^2+[f(t)]^2$

$$\boxed{f(t)=\pm\sqrt{9t^2-16}}$$

3. [25 pts] A curve is described parametrically by:

$$\begin{cases} x(t) &= t^3 - 3t \\ y(t) &= 1 - \dfrac{12}{t} \end{cases}$$

for all t where $x(t)$ and $y(t)$ are well-defined.

I. Find $\frac{dy}{dx}$ in terms of t. Simplify your final answer!

$\frac{dx}{dt} = 3t^2 - 3$, $\frac{dy}{dt} = \frac{12}{t^2}$ so $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12/t^2}{3t^2-3}$

$$= \frac{12}{3t^2(t^2-1)}$$

$$= \boxed{\frac{4}{t^2(t^2-1)}}$$

II. Give the Cartesian equation(s) for any vertical tangent lines to the curve or state that there are none.

We need t such that $\frac{dy}{dx} \to \pm\infty$ but where x, y are defined $\to$ $t^2(t^2-1) = 0$

$t^2(t+1)(t-1) = 0 \to \underline{t = 0, -1, 1}$

Note that $y(0)$ is undefined, so there is no vertical tangent line at $t=0$! (there is an asymptote - graph this!).

The vertical tangents are $x = x(-1)$ and $x = x(1)$

$\boxed{x = 2}$ $\boxed{x = -2}$

III. Give the Cartesian equation(s) for any horizontal tangent lines to the curve or state that there are none.

We need $\frac{dy}{dx} = 0$, but there are no such t values since $\frac{4}{t^2(t^2-1)} \neq 0$!

$\to$ $\boxed{\text{No horizontal asymptotes}}$

IV. Find the Cartesian equation for the tangent line to the curve when $t = 2$.

When $t=2$: $m_{tan} = \left.\frac{dy}{dx}\right|_{t=2} = \frac{4}{(2)^2[(2)^2-1]} = \frac{1}{3}$.

The tangent line is:

$$y - y(2) = m_{tan}[x - x(2)]$$

Noting $x(2) = (2)^3 - 3(2) = 2$

$y(2) = 1 - \frac{12}{(2)} = -5$ gives:

$y - (-5) = \frac{1}{3}[x - 2]$ or $\boxed{y = \frac{1}{3}x - \frac{17}{3}}$

V. Find the Cartesian equations of any horizontal asymptotes to the curve or state that there are none.

Recall $y = c$ is a horizontal asymptote for $y=f(x)$ as $x \to \pm\infty$ if and only if $\lim_{x\to\pm\infty} f(x) = c$.

Here, we see $\lim_{t\to\infty} x(t) = \infty$ and $\lim_{t\to-\infty} x(t) = -\infty$, so

we check $\lim_{t\to\pm\infty} y(t) = \lim_{t\to\pm\infty}\left(1 - \frac{12}{t}\right) = 1$.

Thus, $\boxed{y=1 \text{ is a horizontal asymptote as } x \to \pm\infty}$.

4. [15 pts] Consider the polar curves $r = 2$ and $r = 2 - 2\cos\theta$.

   I. Find the polar form (r, θ) for all of the intersection points of the curves.

   In your answer(s), use values for θ with $0 \le \theta \le 2\pi$.

From the pictures, we expect 2 intersection points. Check first:

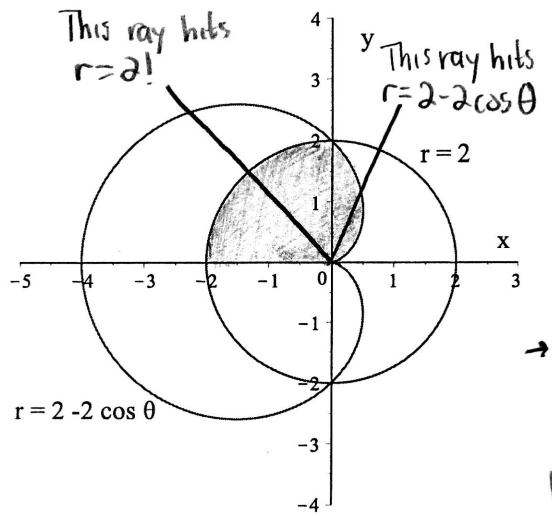
$$2 = 2 - 2\cos\theta$$

$$0 = -2\cos\theta$$

$$\cos\theta = 0 \rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

Since $r = 2$ for both, the intersection pts are $\boxed{(r,\theta) = (2, \tfrac{\pi}{2}),\ (2, \tfrac{3\pi}{2})}$

   II. Set up, but DO NOT EVALUATE, an integral or sum of integrals that gives the area that lies inside of both the circle $r = 2$ and the cardioid $r = 2 - 2\cos\theta$.



By symmetry, the area is twice the area of the shaded region.

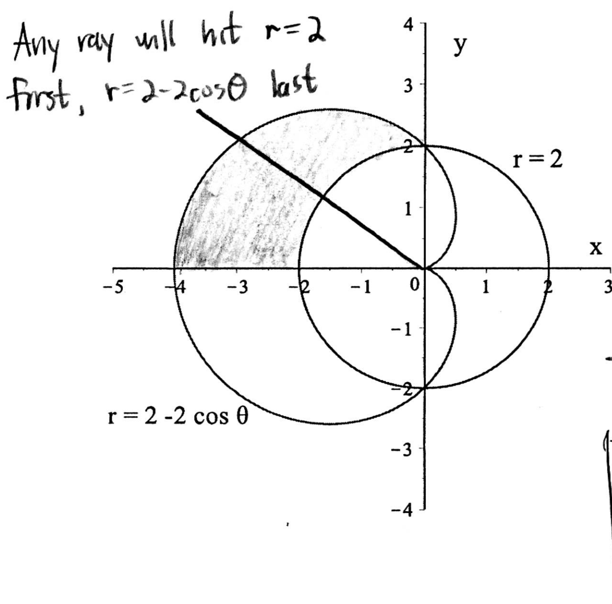
- The outer curve depends on where you draw a ray!

$$\rightarrow A = 2\left[\int_0^{\pi/2} \tfrac{1}{2}(2-2\cos\theta)^2\,d\theta + \int_{\pi/2}^{\pi} \tfrac{1}{2}(2)^2\,d\theta\right]$$

$\uparrow$ symmetry

$$\boxed{A = \int_0^{\pi/2} [2-2\cos\theta]^2\,d\theta + \int_{\pi/2}^{\pi} 2^2\,d\theta}$$

   III. Set up, but DO NOT EVALUATE, an integral or sum of integrals that gives the area that lies inside of the the cardioid $r = 2 - 2\cos\theta$ but outside of the circle $r = 2$.



- By symmetry, the area is twice the area of the shaded region.
- There is a single inner and outer curve!

symmetry

$$\rightarrow A = 2\int_{\pi/2}^{\pi} \tfrac{1}{2}\left[r_{outer}^2 - r_{inner}^2\right]d\theta$$

$$\boxed{A = \int_{\pi/2}^{\pi} [(2-2\cos\theta)^2 - (2)^2]\,d\theta}$$

5. [15 pts] Suppose a curve in three dimensions is given by the vector-valued function:

$$\vec{r}(t) = \langle 3t^2, 4t^2, 2\rangle, 0 \le t \le 1.$$

Determine if the curve is parameterized by arclength. If it is not, find a description of the curve that uses arclength as a parameter.

A curve is parameterized by arclength if and only if $|\vec{r}\,'(t)| = 1$ for <u>every</u> t!

<u>Step 1:</u> Compute $|\vec{r}\,'(t)|$:

$\vec{r}\,'(t) = \langle 6t, 8t, 0\rangle$

$|\vec{r}\,'(t)| = \sqrt{(6t)^2+(8t)^2} = \sqrt{36t^2+64t^2} = \sqrt{100t^2} = \underline{10t}$.

→ The curve is <u>NOT</u> parameterized by arclength!

<u>Step 2:</u> Find $s(t)$ from $s = \int_0^t |\vec{r}\,'(\tau)|\, d\tau$

$s = \int_0^t 10\tau\, d\tau$

$s = 5\tau^2\Big|_0^t$

$\underline{s = 5t^2}$ ← **Note:** s is increasing in t, the curve has length $s(1) = 5$

<u>Step 3:</u> Find t as a function of s.

Here, $\vec{r}(t)$ only has t^2, so we find $\underline{t^2 = \frac{s}{5}}$.

<u>Step 4:</u> Substitute.

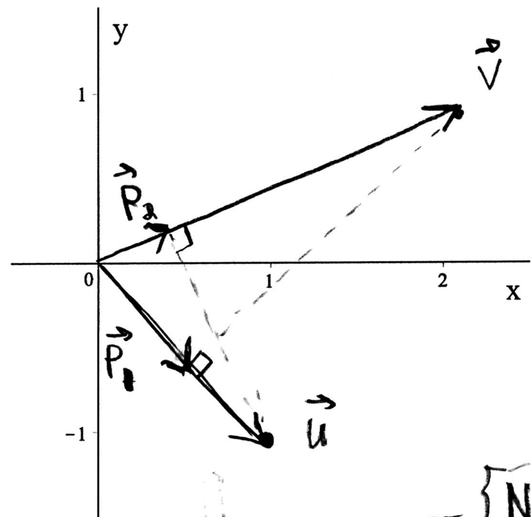
$\vec{r}(t) = \langle 3t^2, 4t^2, 2\rangle, \quad 0 \le t \le 1$

$\vec{r}(s) = \langle 3(\frac{s}{5}), 4(\frac{s}{5}), 2\rangle$

$$\boxed{\vec{r}(s) = \left\langle \frac{3s}{5}, \frac{4s}{5}, 2\right\rangle, \ 0 \le s \le 5}$$

6. [25 pts] Suppose $\vec{u} = \hat{\imath} - \hat{\jmath}$ and $\vec{v} = 2\hat{\imath} + \hat{\jmath}$ are vectors in **two** dimensions.

I. On the axes provided, sketch $\vec{u}$, $\vec{v}$, $\vec{P}_1 = proj_{\vec{u}}\,\vec{v}$, and $\vec{P}_2 = proj_{\vec{v}}\,\vec{u}$.



Note these match the vectors drawn in I!

II. Calculate $\vec{P}_1$ and $\vec{P}_2$.

$$\vec{P}_1 = proj_{\vec{u}}\,\vec{v} = \left[\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\right]\vec{u} = \left[\frac{2(1)+(1)(-1)}{1(1)+(-1)(-1)}\right](\hat{\imath}-\hat{\jmath})$$

$$\boxed{\vec{P}_1 = \frac{1}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath}}$$

$$\vec{P}_2 = proj_{\vec{v}}\,\vec{u} = \left[\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right]\vec{v} = \left[\frac{(1)(2)+(-1)(1)}{(2)(2)+(1)(1)}\right](2\hat{\imath}+\hat{\jmath})$$

$$\boxed{\vec{P}_2 = \frac{2}{5}\hat{\imath} + \frac{1}{5}\hat{\jmath}}$$

III. Let $\vec{w}_1$ be a vector parallel to $\vec{v}$. Show that $proj_{\vec{w}_1}\,\vec{u} = \vec{P}_2$.

This statement should be obvious geometrically! To show it analytically, note since $\vec{w}_1$ is parallel to $\vec{v}$, there is a constant c such that $\vec{w}_1 = c\vec{v} = c\langle 2, 1\rangle = \langle 2c, c\rangle$

$$proj_{\vec{w}_1}\,\vec{u} = \left[\frac{\vec{u}\cdot\vec{w}_1}{\vec{w}_1\cdot\vec{w}_1}\right]\vec{w}_1 = \left[\frac{(1)(2c)+(-1)(c)}{(2c)(2c)+(c)(c)}\right](2c\,\hat{\imath} + c\hat{\jmath})$$

$$= \frac{c}{5c^2}\cdot c\,[2\hat{\imath}+\hat{\jmath}]$$

$$= \frac{2}{5}\hat{\imath} + \frac{1}{5}\hat{\jmath} = \vec{P}_2 \checkmark$$

Now, suppose that $\vec{u} = \hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{v} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$ are vectors in **three** dimensions.

IV. Find a vector $\vec{w}_2$ of magnitude 4 that is orthogonal to both $\vec{u}$ and $\vec{v}$.

- A vector orthogonal to both $\vec{u}, \vec{v}$ is:

$$\vec{w} = \vec{u} \times \vec{v} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} = -3\hat{\imath} + 0\hat{\jmath} + 3\hat{k}$$

but $|\vec{w}| = \sqrt{(-3)^2 + 3^2} = \sqrt{18}$ or $3\sqrt{2}$.

- $\hat{w} = \frac{\vec{w}}{|\vec{w}|} = \frac{1}{3\sqrt{2}} \langle -3, 0, 3 \rangle$. is a unit vector in the correct direction, so $4\hat{w}$ is a vector of length 4 in the appropriate direction:

$$\vec{w}_2 = 4\hat{w} = 4 \cdot \frac{1}{3\sqrt{2}} \langle -3, 0, 3 \rangle = \frac{\overset{2}{\cancel{4}}\sqrt{2}}{3 \cdot \cancel{2}} \langle -3, 0, 3 \rangle$$

$$\boxed{\vec{w}_2 = (-2\sqrt{2})\hat{\imath} + (2\sqrt{2})\hat{k}}$$

V. Verify that $\vec{w}_2$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

We check:

i) $\vec{w}_2 \cdot \vec{u} = (-2\sqrt{2})(1) + 0(-1) + (2\sqrt{2})(1) = 0$ ✓

→ So $\vec{w}_2 \perp \vec{u}$!

ii) $\vec{w}_2 \cdot \vec{v} = (-2\sqrt{2})(1) + 0(2) + (2\sqrt{2})(1) = 0$ ✓

→ So $\vec{w}_2 \perp \vec{v}$!

Answers:

1. **Multiple choice**

   I. C.

   II. B.

   III. B.

2. **Short Answer**

   I. Domain: $\{(x,y)|16-2x^2+y^2 \geq 0\}$

   II. Range: $[0,\infty)$

   III. xz-trace is $z=\sqrt{16-2x^2}$

   IV. $2x^2=y^2$

   V. 3

   VI. $f(t)=\pm\sqrt{9t^2-16}$

3. I. $\dfrac{dy}{dx}=\dfrac{4}{t^4-t^2}$

   II. Vertical Tangent Lines: $x=-2$, $x=2$

   III. No horizontal tangent lines

   IV. $y=\dfrac{1}{3}x-\dfrac{17}{3}$

   V. Horizontal asymptotes are $y=1$ as $x\to\pm\infty$.

4. I. $(r,\theta)=\left(2,\dfrac{\pi}{2}\right),\left(2,\dfrac{3\pi}{2}\right)$

   II. $\displaystyle\int_0^{\pi/2}(2-2\cos\theta)^2\,d\theta+\int_{\pi/2}^{\pi}2^2\,d\theta$

   III. $\displaystyle\int_{\pi/2}^{\pi}\left[(2-2\cos\theta)^2-2^2\right]\,d\theta$

5. The curve is not parameterized by arclength since $|\vec{r}\,(t)|\neq 1$ for all t.

   A parameterization that uses arclength is $\vec{r}\,(s)=\left\langle\dfrac{3s}{5},\dfrac{4s}{5},2\right\rangle, 0\leq s\leq 5.$

6. I. See solutions

   II. $\vec{P}_1=\dfrac{1}{2}\hat{\imath}-\dfrac{1}{2}\hat{\jmath}$ and $\vec{P}_2=\dfrac{2}{5}\hat{\imath}+\dfrac{1}{5}\hat{\jmath}$

   III. See solutions

   IV. $\vec{w}_2=\left(-2\sqrt{2}\right)\hat{\imath}+\left(2\sqrt{2}\right)\hat{k}$

   V. See solutions