Gordon Prize Examination

February 22, 1997

1. Does this series converge?

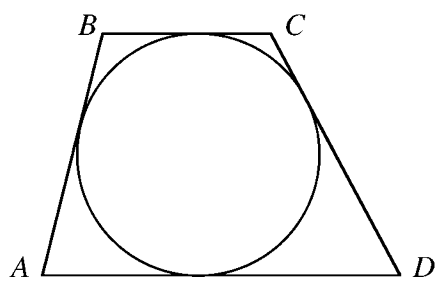
$$1 + \frac{1 \cdot 3}{2} + \frac{2!\,3^2}{3^2} + \frac{3!\,3^3}{4^3} + \frac{4!\,3^4}{5^4} + \cdots + \frac{n!\,3^n}{(n+1)^n} + \cdots$$

2. Prove that $1997^{13} - 1997^{11} - 1997^9 + 1997^5 + 1997^3 - 1997$ is divisible by 105.

3. Suppose $1 < a_1 < a_2 < \cdots < a_n$ are positive integers. Prove:

$$\left(1 - \frac{1}{a_1^2}\right)\left(1 - \frac{1}{a_2^2}\right)\left(1 - \frac{1}{a_3^2}\right)\cdots\left(1 - \frac{1}{a_n^2}\right) > \frac{1}{2}.$$

4. Given: $ABCD$ is circumscribed around the circle, and AD is parallel to BC. Write T for the area of $ABCD$. Prove that $AB + CD \geq 2\sqrt{T}$.



5. Let C be a circle in the xy-plane. Suppose its center has both coordinates irrational. Can one inscribe in C a triangle whose vertices all have both coordinates rational?

6. Let $k, a_1, a_2, \cdots, a_n$ be integers. Prove that the determinant

$$\begin{vmatrix} a_1^2 + k & a_1 a_2 & a_1 a_3 & \cdots & a_1 a_n \\ a_2 a_1 & a_2^2 + k & a_2 a_3 & \cdots & a_2 a_n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & a_n a_2 & a_n a_3 & \cdots & a_n^2 + k \end{vmatrix}$$

is divisible by k^{n-1}.

You may take this sheet with you.
Be sure to hand in separately the cover sheet
(with your name, rank, student number, and secret code name).
Put your secret code name at the top of each answer sheet.