Gordon Prize Examination

February 23, 2002

1. If the series $\sum_{n=0}^{\infty} a_n$ converges, must $\sum_{n=0}^{\infty} a_n^3$ also converge?

2. Let S be a set of fifteen integers contained in the interval $[2, 2002]$ so that every two of them are relatively prime. (Two positive integers are said to be *relatively prime* if they have no common divisor greater than 1.) Show that S contains a prime number.

3. Let T be a triangle with sides a, b, c and angle γ opposite side c. The area A and the angle γ are known. Prove that there exist a and b for which c is minimal, and find them.

4. Let n be a positive integer and $A_1, \ldots, A_{2n}$ be distinct points in the plane. Show that there are at least n line segments, none of which crosses another, connecting pairs of points from $A_1, \ldots, A_{2n}$.

5. Can the plane be completely covered by 2002 infinite sectors such that sum of their angles is less than 360°? (An *infinite sector* of angle α is the unbounded region between two rays making angle α, as shown.)

6. Suppose $0, 1, u$ form an equilateral triangle in the complex plane. The set of all complex numbers $m + nu$, with m and n integers, forms an equilateral triangle grid. Consider the subset L of all complex numbers $m + nu$ where m, n are non-negative integers such that $m + n \leq 2002$. Suppose that more than two-thirds of the points in L are colored blue. Show there are at least three blue points in L that are the vertices of an equilateral triangle.