Gordon Prize Examination

February 22, 2003

1. Let $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-2} + F_{n-1}$ for $n = 3, 4, \ldots$ be the Fibonacci sequence. Show that 2003 divides F_n for some n.

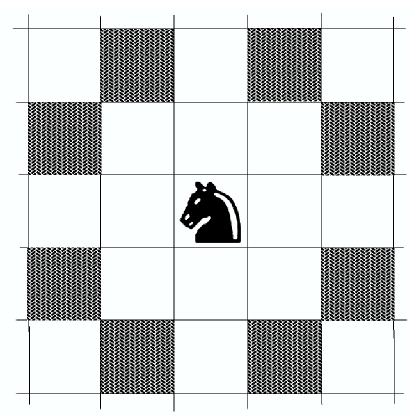
2. Find the maximum of $f(x) = (1-x)(1-y)(1-z)$ over all nonnegative x, y, z with $x^2 + y^2 + z^2 = 1$.

3. Prove that $\sin\left(n^2\right)$ does not converge to 0 as $n \to \infty$. [The limit is taken over positive integers n.]

4. Show that no angle of a triangle whose vertices are lattice points in the plane can be equal to $15°$. [A *lattice point* is a point (x, y) where x and y are integers.]

5. Some numbers can be expressed as an alternating sum of an increasing sequence of distinct powers of 2. For example, $1 = -1+2$; $2 = -2+4$; $3 = 1-2+4$; $4 = -4+8$; $5 = 1-4+8$; $6 = -2+8$; etc. Is every positive integer expressible in this fashion?

6. Consider a 4×2003 chessboard. Is it possible for a chess knight to follow a path that lands on each of the squares exactly once and then returns to the starting square? [A *knight* is a chess piece that moves as shown: from the center square it can move to any of the 8 shaded squares.]



You may take this sheet with you.
Be sure to hand in separately the cover sheet
(with your name, rank, student number, and secret code name).
Put your secret code name at the top of each answer sheet.