Gordon Prize Examination

February 28, 2004

1. Given any selection of 1004 distinct integers from the set $\{1, 2, \dots, 2004\}$, show that some three of the selected integers have the property that one is the sum of the other two.

2. What is the greatest integer less than or equal to $\dfrac{1}{e^{\frac{1}{2004}} - 1}$?

3. Let A, B be $n \times n$ matrices with real coefficients such that A is invertible. Is it possible that $AB - BA = A$?

4. Show that, for $a > 0$, $\displaystyle\int_{\frac{1}{2004}}^{2004} \frac{x - 1}{1 + ax + ax^2 + x^3}\, dx = 0.$

5. Let a, b and c be complex numbers forming a triangle in the complex plane. Show that there is a complex number p such that all of the following numbers are real:

$$\frac{(a-b)(a-c)}{(a-p)^2}, \qquad \frac{(b-a)(b-c)}{(b-p)^2}, \qquad \frac{(c-a)(c-b)}{(c-p)^2}.$$

6. Let α, β, γ be the angles of a triangle. Show that $\cos\alpha \cdot \cos\beta \cdot \cos\gamma \leq \dfrac{1}{8}$.

Answer each problem on the separate sheet containing the statement of the problem, or on an additional sheet. "Answers" without appropriate reasoning are insufficient. **Put your secret code name at the top of each answer sheet.** Please write on only one side of the answer sheets.

You may take this sheet with you when you leave. Be sure to hand in separately the cover sheet (with your name, rank, student number, and secret code name).