Gordon Prize Exam

February 18, 2006

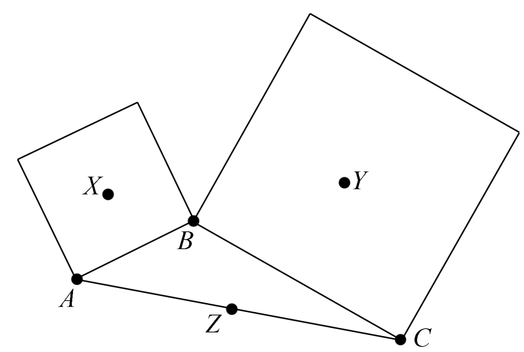
1. There is an integer $N > 100$ such that N is a square, the last digit of N (in base ten) is not 0, and when the last two digits are deleted, the result is still a square. What is the largest N with this property?

2. Let $f\colon (a,b) \to \mathbb{R}$ be twice continuously differentiable, and assume $f''(x) \neq 0$ for all $x \in (a,b)$. Prove that two chords on the graph of f cannot bisect each other. (A **chord** on the graph is a line segment that joins two points on the graph.)

3. Let $A = (a_{ij})$ be a 2006×2006 "checkerboard" matrix of 0s and 1s. That is, $a_{ij} = 0$ if $i + j$ is even and $a_{ij} = 1$ if $i + j$ is odd. Compute the characteristic polynomial of A.

4. A sequence $\{a_n\}$ of **positive** real numbers satisfies $a_0 = 1$ and $a_{n+2} = 2a_n - a_{n+1}$ for $n \geq 0$. (Note that a_1 is not specified.) Find a_{2006}; justify your answer.

5. Let ABC be a triangle in the plane. Erect squares externally on its sides AB and BC. Let X and Y be the centers of these squares and let Z be the midpoint of CA. Prove that the triangle XYZ is an isosceles right triangle. (It may help to use complex numbers.)



6. For each integer $k > 1$, let r_k be the remainder when 2^{1003} is divided by k. Prove that $r_2 + r_3 + \cdots + r_{1003} > 2006$.

Answer each problem on the separate sheet containing the statement of the problem, or on an additional sheet. "Answers" without appropriate reasoning are insufficient. **Put your secret code name at the top of each answer sheet.** Please write on only one side of the answer sheets, and inside the frame.

You may take this sheet with you when you leave. Be sure to hand in separately the cover sheet (with your name, rank, student number, and secret code name).