Rasor-Bareis Prize Examination

February 23, 2002

1. Row k of Pascal's triangle contains the coefficients in the expansion of $(x+y)^k$. (So that, for example, $1, 4, 6, 4, 1$ is "row 4".) Which row is the first one that contains a number divisible by 2002?

2. The projections of a body in 3-dimensional space on two planes are disks. Prove that the disks have equal radii.

3. Let A, B, C, D be the corners of a rectangular billiard table of size $M \times N$, where M and N are positive integers. A billiard ball (that is, a point) is sent from the corner A at the angle $30°$ to the side AB of the table. When the ball reaches a side of the table, it bounces off with angle of reflection equal to angle of incidence; if the ball exactly hits a corner, it bounces directly back retracing its path. Prove that the ball will never get back to the corner A.

4. Let n be a positive integer and $A_1, \ldots, A_{2n}$ be distinct points in the plane. Show that there are at least n line segments, none of which crosses another, connecting pairs of points from $A_1, \ldots, A_{2n}$.

5. Can the plane be completely covered by 2002 infinite sectors such that sum of their angles is less than $360°$? (An *infinite sector* of angle α is the unbounded region between two rays making angle α, as shown.)

6. Suppose $0, 1, u$ form an equilateral triangle in the complex plane. The set of all complex numbers $m + nu$, with m and n integers, forms an equilateral triangle grid. Consider the subset L of all complex numbers $m + nu$ where m, n are non-negative integers such that $m + n \leq 2002$. Suppose that more than two-thirds of the points in L are colored blue. Show there are at least three blue points in L that are the vertices of an equilateral triangle.