Rasor-Bareis Prize Examination

February 22, 2003

1. Let $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-2} + F_{n-1}$ for $n = 3, 4, \ldots$ be the Fibonacci sequence. Show that 2003 divides F_n for some n.

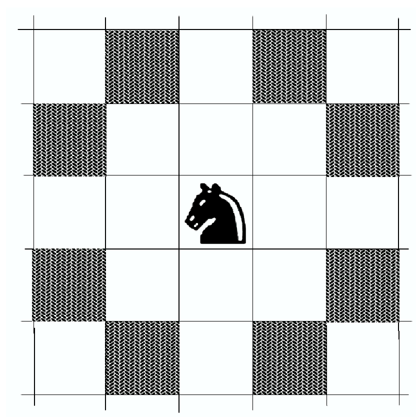
2. Let α, β, γ be the angles in a triangle. Show that if $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, then it is a right triangle.

3. Find all integer solutions of $x^3 - 2y^3 - 4z^3 = 0$.

4. Prove that if a, b, c are complex numbers of absolute value 1, then $|ab + ac + bc| = |a + b + c|$.

5. Show that $\lim\limits_{n\to\infty} n \sin\big(2\pi e n!\big)$ exists, and compute its value. [The limit is taken over positive integers n, and $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.]

6. Consider a 4×2003 chessboard. Show that it is not possible for a chess knight to follow a path that lands on each of the squares exactly once and then returns to the starting square. [A *knight* is a chess piece that moves as shown: from the center square it can move to any of the 8 shaded squares.]



You may take this sheet with you.
Be sure to hand in separately the cover sheet
(with your name, rank, student number, and secret code name).
Put your secret code name at the top of each answer sheet.