Rasor-Bareis Prize Examination

February 28, 2004

1. Find the value of $\lim_{n\to+\infty} \dfrac{1+2^2+\cdots+n^n}{n^n}$ if it exists. If it does not exist, say why.

2. Let Q be a convex quadrilateral in the plane. Show that a line can be constructed, using straight-edge and compass only, that divides Q into two regions of equal area.

3. Let f be a real-valued function such that $f(2003) = 2\pi$ and $|f(x)-f(y)|^2 \leq |x-y|^3$ for all real numbers x and y. Compute $f(2004)$.

4. Let $P(x)$ be a nonconstant polynomial with integer coefficients. Is it possible that $P(n)$ is a prime number for all integers n?

5. Given any selection of 1004 distinct integers from the set $\{1, 2, \ldots, 2004\}$, show that some three of the selected integers have the property that one is the sum of the other two.

6. Let α, β, γ be the angles of a triangle. Show that $\cos\alpha \cdot \cos\beta \cdot \cos\gamma \leq \dfrac{1}{8}$.

Answer each problem on the separate sheet containing the statement of the problem, or on an additional sheet. "Answers" without appropriate reasoning are insufficient. **Put your secret code name at the top of each answer sheet.** Please write on only one side of the answer sheets.

You may take this sheet with you when you leave. Be sure to hand in separately the cover sheet (with your name, rank, student number, and secret code name).