Rasor-Bareis Prize Exam

February 18, 2006

1. Prove that, given a rectangle R of area 1, one can place nonoverlapping disks inside R so that the sum of their radii is 2006.

2. Let a be a complex number and n a positive integer. Assume $a^n = 1$ and $(a+1)^n = 1$. Show n is a multiple of 6 and $a^3 = 1$.

3. Let f be a function from reals to reals. Assume that $2f(x) + f(1-x) = x + 4$ for all x. Determine the function f.

4. There is an integer $N > 100$ such that N is a square, the last digit of N (in base ten) is not 0, and when the last two digits are deleted, the result is still a square. What is the largest N with this property?

5. Let T be a triangle in the plane, and let P be a parallelogram that lies inside T. Show that the area of P is at most half the area of T.

6. Let $f\colon (a, b) \to \mathbb{R}$ be twice continuously differentiable, and assume $f''(x) \neq 0$ for all $x \in (a, b)$. Show that two chords on the graph of f cannot bisect each other. (A **chord** on the graph is a line segment that joins two points on the graph.)

Answer each problem on the separate sheet containing the statement of the problem, or on an additional sheet. "Answers" without appropriate reasoning are insufficient. **Put your secret code name at the top of each answer sheet.** Please write on only one side of the answer sheets, and inside the frame.

You may take this sheet with you when you leave. Be sure to hand in separately the cover sheet (with your name, rank, student number, and secret code name).