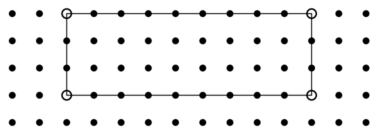
2013 Rasor-Bareis Prize examination problems

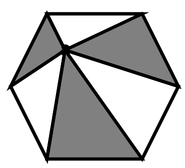
1. Consider an infinite arithmetical progression of positive integers. Prove that there are infinitely many terms in this progression the sum of whose decimal digits is the same.

2. The integer points in the plane (the points with coordinates (n, m), where n and m are integers) are colored with 2013 different colors. Prove that there is a rectangle $\{(n_1, m_1), (n_2, m_1), (n_1, m_2), (n_2, m_2)\}$ whose vertices have the same color.



3. Suppose that $I_1, \ldots, I_n$ are subintervals of $[0, 1]$ such that $\sum_{i=1}^{n} |I_i| \geq 2013$ (where $|I|$ denotes the length of an interval I). Prove that there exists a point $x \in [0, 1]$ that belongs to at least 2013 of the intervals I_i.

4. A point inside a regular 6-gon is connected by straight line segments with the vertices, forming six triangles, which are alternately colored black and white as in the figure below. Prove that the sum of the areas of the black triangles is equal to the sum of the areas of the white triangles.



5. If $x + x^{-1}$ is an integer, prove that $x^{2013} + x^{-2013}$ is also an integer.

6. Let $f\colon \mathbb{R} \longrightarrow (0, \infty)$ be a continuous periodic function having period 1; prove that $\int_0^1 \frac{f(x)dx}{f(x+1/2)} \geq 1$.