Solutions: 2003 Gordon Prize Examination

1. [See the Rasor-Bareis solutions.]

2. [The interesting thing about this problem is that $x = y = z = 1/\sqrt{3}$ is *not* the maximum; instead the maximum occurs on the boundary of the surface.]
Let $g(x,y) = f\big(x, y, \sqrt{1-x^2-y^2}\big)$. Then we are to maximize $g(x,y)$ in the quarter-circle Q defined by $x^2+y^2 \le 1, x \ge 0, y \ge 0$. First consider the interior of Q. Then

$$\frac{\partial g}{\partial x}(x,y) = \frac{(y-1)\left(\sqrt{1-x^2-y^2} - 1 - x + 2x^2 + y^2\right)}{\sqrt{1-x^2-y^2}},$$

$$\frac{\partial g}{\partial y}(x,y) = \frac{(x-1)\left(\sqrt{1-x^2-y^2} - 1 - y + 2y^2 + x^2\right)}{\sqrt{1-x^2-y^2}}.$$

Both derivatives are zero (and the point is in the interior of Q) when $(x,y) = (1/3, 2/3), (1/\sqrt{3}, 1/\sqrt{3}), (2/3, 1/3)$, or $(2/3, 2/3)$. The values of g at these points are $2/27, 2 - 10\sqrt{3}/9, 2/27, 2/27$, respectively. Next consider the boundary segment $y = 0, 0 \le x \le 1$. Now if $h(x) = f(x, 0, \sqrt{1-x^2}) = (1-x)\big(1-\sqrt{1-x^2}\big)$, then $h'(x) = 0$ only at $x = 0$ and $x = 1/\sqrt{2}$. The value of h at these points is $0, 3/2 - \sqrt{2}$, respectively. The other curves are $y = 0$ and $z = 0$ and provide solutions symmetric with these. Comparing all these values, we see the maximum is $3/2 - \sqrt{2}$ and occurs at $(x,y,z) = (1/\sqrt{2}, 1/\sqrt{2}, 0)$, $(1/\sqrt{2}, 0, 1/\sqrt{2})$, $(0, 1/\sqrt{2}, 1/\sqrt{2})$.

3. Claim: $\sin(n^2)$ does not approach 0 as $n \to \infty$. Assume for a contradiction that it does. Then also $\sin\big((n+1)^2\big) \to 0$. Now

$$\sin(2n+1) = \sin\big((n+1)^2 - n^2\big) = \sin\big((n+1)^2\big)\cos(n^2) - \cos\big((n+1)^2\big)\sin(n^2),$$

and therefore $\sin(2n+1) \to 0$ since cosine is bounded. As before: $\sin(2n-1) \to 0$ and then

$$\sin(2) = \sin\big((2n+1) - (2n-1)\big) = \sin(2n+1)\cos(2n-1) - \cos(2n+1)\sin(2n-1),$$

so $\sin(2) \to 0$. But 2 is not an integer multiple of π, so $\sin(2) \ne 0$, and therefore certainly does not converge to 0 as $n \to \infty$. This contradiction completes the proof that $\sin(n^2)$ does not approach 0.

4. [Solution I by Ben Przybyla]
A line-segment where both endpoints are lattice points has rational slope (except for a vertical segment). Suppose A, B, C are lattice points and $\angle ABC = 15°$. If either AB or BC is vertical, reflect the whole picture about the line $x = y$ to obtain another lattice-triangle where neither AB nor BC is vertical. The slopes of both AB and BC are rational, say u and v. Then $uv \neq -1$ (since that would mean the lines are perpendicular), so from the formula

$$\arctan(u) - \arctan(v) = \arctan\left(\frac{u-v}{1+uv}\right)$$

we conclude that $\tan(15°)$ would be rational. But $\tan(15°) = \tan(60° - 45°) = 2 - \sqrt{3}$ is, in fact, irrational. So there is no such lattice triangle.

[Solution II]
Suppose that A, B, C are lattice points and $\angle BAC = 15°$. By the Law of Cosines,

$$\cos 15° = \frac{|AB|^2 + |AC|^2 - |BC|^2}{2|AB||AC|}.$$

Since A, B, C are lattice points, $|AB|^2$, $|BC|^2$ and $|AC|^2$ are integers, and hence

$$\cos^2 15° = \frac{\left(|AB|^2 + |AC|^2 - |BC|^2\right)^2}{4|AB|^2|AC|^2}$$

is a rational number. It follows that $\cos 30° = 2\cos^2 15° - 1$ is rational, which is wrong.

[Note: Rotation does not preserve lattice points, so one cannot begin by rotating the triangle to make one side horizontal.]

5. [Solution I by Charles Estill]
We claim that any positive integer n can be written in the form specified. Write n in base 2, then write $2n$ in base 2 by appending a 0 on the right. Now subtract $2n - n$ using these binary representations: When there is a 1 in the same place in both representations, they cancel, and what remains is exactly an alternating sum of powers of 2.

[Solution II]
Proof by induction: Assume that for some $n \in \mathbb{N}$ all positive integers that are less than 2^n are representable in the form $2^{n_1} - 2^{n_2} + \cdots \pm 2^{n_k}$ with $n \geq n_1 > n_2 > \cdots > n_k \geq 0$, $k \geq 2$. Now let $2^n \leq m < 2^{n+1}$. If $m = 2^n$, then $m = 2^{n+1} - 2^n$; and if $m > 2^n$, then $2^{n+1} - m < 2^n$, so by the induction hypothesis $2^{n+1} - m = 2^{n_1} - 2^{n_2} + \cdots \pm 2^{n_k}$, and thus $m = 2^{n+1} - 2^{n_1} + 2^{n_2} - \cdots \mp 2^{n_k}$.

6. Suppose there is such a closed knight path landing on all the squares. Let's say the board consists of 4 "columns" and 2003 "rows". In each row there are two "outer" squares on the ends and two "inner" squares between. Note that a knight can move from an outer square only to an inner square. Since there are an equal number of outer and inner squares, a closed path that covers them all exactly once must therefore alternate between inner and outer squares. On the other hand, we may imagine the standard coloring of the board by alternating black and white squares. A knight can move from a black square only to a white square (and from a white square only to a black square). So, in our closed path, all the outer squares of the path must be the same color (and all the inner squares of the path the opposite color). But half of the outer squares are of each color, so we obtain a contradiction.