Solutions: 2005 Gordon Prize Examination

1. We may assume that $b_1, \ldots, b_n \neq 0$. For all $x \neq 0$, we have

$$\left| a_1 b_1 \frac{\sin(b_1 x)}{b_1 x} + a_2 b_2 \frac{\sin(b_2 x)}{b_2 x} + \ldots + a_n b_n \frac{\sin(b_n x)}{b_n x} \right| \leq \left| \frac{\sin x}{x} \right|.$$

Observe that $\lim_{h \to 0} \frac{\sin h}{h} = 1$. Taking the limit as $x \to 0$ of both sides, we get $\left| a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \right| \leq 1$.

2. The solution set is $\{0, 1\}$.

Proof. Let $x \in \mathbb{R}$ satisfy $2003^x - 2002^x = 2005^x - 2004^x$. By the Mean Value Theorem, applied to the function $f(t) = t^x$, there exists $a \in [2002, 2003]$ such that $2003^x - 2002^x = xa^{x-1}$. Similarly, there exists $b \in [2004, 2005]$ such that $2005^x - 2004^x = xb^{x-1}$. Then $xa^{x-1} = xb^{x-1}$ and $a < b$, so either $x = 0$ or $x = 1$.

3. Perform the following row operations on A: subtract or add the first row of A from/to each other row in order to "kill" all elements of the first column, except for the first one, $A = \begin{pmatrix} \pm 1 & \pm 1 & \ldots & \pm 1 \\ \pm 1 & \pm 1 & \ldots & \pm 1 \\ \vdots & \vdots & & \vdots \\ \pm 1 & \pm 1 & \ldots & \pm 1 \end{pmatrix} \mapsto A' = \begin{pmatrix} \pm 1 & \pm 1 & \ldots & \pm 1 \\ 0 & & & \\ \vdots & & \text{B} & \\ 0 & & & \end{pmatrix}$. Then $\det A' = \det A$, and each entry of the matrix B is equal to either 0, or 2, or -2. Expanding by minors along the first column yields $\det A' = \pm \det B$. Since the entries of the $(n-1) \times (n-1)$-matrix B are all even, $\det B$ is divisible by 2^{n-1}.

4. Put $z = e^{-i\pi/3}$. Then $C = A + z(B - A)$ and $Z = X + z(Y - X)$, so

$$\begin{aligned} C + Z = A + z(B - A) + X + z(Y - X) &= (A + X) + z(B - A + Y - X) \\ &= (A + X) + z[(B + Y) - (A + X)]. \end{aligned}$$

Note that $C + Z = A + X$ if and only $B + Y = A + X$.

5. Solution 1. Observe that

$$\begin{aligned} (\sqrt{65} + 8)^{2005} &= \sum_{n=0}^{2005} \binom{2005}{n} \sqrt{65}^n 8^{2005-n} \\ &= \sum_{k=0}^{1002} \binom{2005}{2k} \sqrt{65}^{2k} 8^{2005-2k} + \sum_{k=0}^{1002} \binom{2005}{2k+1} \sqrt{65}^{2k+1} 8^{2005-(2k+1)} \end{aligned}$$

and

$$\begin{aligned} (\sqrt{65} - 8)^{2005} &= \sum_{n=0}^{2005} \binom{2005}{n} \sqrt{65}^n (-8)^{2005-n} \\ &= -\sum_{k=0}^{1002} \binom{2005}{2k} \sqrt{65}^{2k} 8^{2005-2k} + \sum_{k=0}^{1002} \binom{2005}{2k+1} \sqrt{65}^{2k+1} 8^{2005-(2k+1)}. \end{aligned}$$

The first term in the right hand side of both equalities is an integer and the last terms coincide, thus the fractional part of $(8+\sqrt{65})^{2005}$ equals the fractional part of $(\sqrt{65}-8)^{2005}$. Now, since

$$8.1^2 = (8+0.1)^2 = 64 + 1.6 + 0.01 > 65,$$

we have $8 < \sqrt{65} < 8.1$, and so $0 < \sqrt{65} - 8 < 0.1$. It follows that $0 < (\sqrt{65}-8)^{2005} < 10^{-2005}$, which means that the first 2005 digits of the decimal expansion of $(\sqrt{65}-8)^{2005}$ are zero.

Solution 2. Let $s = 8 + \sqrt{65}$ and $r = 8 - \sqrt{65}$. Then $-0.1 < r < 0 < s$. Also, r and s are the roots of the polynomial $x^2 - 16x - 1$. Hence r^n and s^n both satisfy the recurrence $f(n+2) = 16f(n+1) + f(n)$. Hence $r^n + s^n$ also satisfies this recurrence, with $f(0) = 2$ and $f(1) = 16$, so $r^n + s^n$ is an integer for all $n \geq 0$. Thus $r^{2005} + s^{2005} = k$, for some positive integer k. Therefore,

$$s^{2005} = k - r^{2005} = k + (-r)^{2005}.$$

Since $0 < -r < 0.1$, it follows that s^{2005} has at least 2005 zeros after the decimal point.

6. (Same as RB6.) For each $n \geq 1$, there are at most $n^{1/2}$ squares and at most $n^{1/3}$ cubes contained in $\{1, \ldots, n\}$, so there are at most $n^{1/2} \cdot n^{1/3} = n^{5/6}$ numbers of the form $x^2 + y^3$ contained in $\{1, \ldots, n\}$, where x and y are nonnegative integers. Since $n - n^{5/6} \longrightarrow \infty$ as $n \longrightarrow \infty$, the result follows.