Gordon solutions

1. A positive integer is called a *palindrome* if its base-10 expansion is unchanged when it is reversed. For example, 121 and 7447 are palindromes. Show that if we denote by p_n the nth smallest palindrome, then $\sum_{n=1}^{\infty} \frac{1}{p_n}$ converges.

Proof: Every term of this sum is positive, so showing that the sum is bounded suffices to show convergence. Any palindrome whose base-10 expansion has k digits is uniquely determined by the first $\lceil \frac{k}{2} \rceil$ digits of its base-10 expansion. Therefore, there are at most $10^{\lceil k/2 \rceil} \leq 10^{1+k/2}$ such palindromes for any k. For any positive integer k, denote by A_k the set of palindromes whose base-10 expansion have k digits. Then, any element of A_k is greater than or equal to 10^{k-1}, and A_k has at most $10^{1+k/2}$ members for every k. This implies that

$$\sum_{n=1}^{\infty} \frac{1}{p_n} = \sum_{k=1}^{\infty} \left(\sum_{p \in A_k} \frac{1}{p} \right) \leq \sum_{k=1}^{\infty} 10^{1+k/2} \frac{1}{10^{k-1}} = \sum_{k=1}^{\infty} 10^{2-k/2} = 100 \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{10}} \right)^k,$$

which is a geometric series with ratio less than one, and so converges.

2. Let $z_1, ..., z_{2007}$ be equally spaced points on the unit circle. Prove that

$$\|z_1 - z_2\| \, \|z_1 - z_3\| \cdots \|z_1 - z_{2007}\|,$$

the product of the lengths of the chords from z_1 to all of the other z_k, is equal to 2007.

Solution. In the complex plane, we may assume the points z_k are equally spaced on the circle $|z| = 1$ and $z_1 = 1$. These 2007 complex numbers are the zeros of the polynomial $z^{2007} - 1$, so $z_2, \cdots, z_{2007}$ are the zeros of the polynomial

$$\frac{z^{2007} - 1}{z - 1} = z^{2006} + z^{2005} + \cdots + z^2 + z + 1.$$

Call this polynomial $p(z)$. Because we know all the complex zeros, we know a factorization:

$$p(z) = (z - z_2)(z - z_3) \cdots (z - z_{2007}).$$

Substitute $z = 1$ to get:

$$(1 - z_2)(1 - z_3) \cdots (1 - z_{2007}) = p(1) = 1^{2006} + 1^{2005} + \cdots + 1^2 + 1^1 + 1 = 2007.$$

So its absolute value (which is what we must compute) is also 2007.

3. Show that integers a, b, c do not exist such that $a+b+c = -45$ and $ab+bc+ca = 9$.

Solution I.

Since -45 is odd, either none, or exactly two of the integers a, b, c can be even. If two of them were even, then $ab + bc + ca$ would be even, and would not be 9. Hence all three integers must be odd. So they are either 1 or 3 mod 4. But then $ab + bc + ca$ can only be one of:

$$\begin{aligned}
1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 &= 3 \pmod 4, \\
1 \cdot 1 + 1 \cdot 3 + 3 \cdot 1 &= 3 \pmod 4, \\
1 \cdot 3 + 3 \cdot 3 + 3 \cdot 1 &= 3 \pmod 4, \\
3 \cdot 3 + 3 \cdot 3 + 3 \cdot 3 &= 3 \pmod 4.
\end{aligned}$$

It cannot be 9.

Solution II.

From the given equations it follows that $a^2+b^2+c^2 = (a+b+c)^2-2(ab+bc+ca) = (-45)^2 - 2 \times 9 = 2007$. Now 2007 is 7 modulo 8. The squares modulo 8 are 0, 1, and 4, and no three of these can add to 7 modulo 8. So this is impossible.

4. Let W be a polynomial with real coefficients. Assume that $W(x) \geq 0$ for all real x. Prove that W may be written as a sum of squares of polynomials.

Solution I.

The proof proceeds by induction on the degree of $W(x)$. If $\deg W = 0$, then the assertion holds, as any non-negative number is a square. Note that if $\deg W$ is odd then $W(x)$ is not positive everywhere.

Suppose $\deg W = n$. Its infimum is nonnegative, hence a square: say $m^2 = \inf_x W(x) = W(a)$ for some a (the infimum is attained as $W(x)$ is a polynomial of even degree). Thus $W(x) = m^2 + (x-a)^k Q(x)$ with $\deg Q < \deg W$ and $(x-a)^k Q(x) \geq 0$ for all x.

If k were odd the expression $(x-a)^k Q(x)$ would change sign at a; so k is even. Thus $Q(x)$ is nonnegative, and by inductive assumption it is a sum of squares. Therefore $W(x) = m^2 + (x-a)^k Q(x)$ is a sum of squares.

Solution II.

Every polynomial with real coefficients factors over the complex numbers as

$$W(x) = \prod_i (x - a_i)^{m_i} \prod_j [(x - q_j)(x - \overline{q_j})]^{n_i},$$

where the a_i are (zero or more) distinct real zeros, and the $q_j, \overline{q_j}$ are (zero or more) distinct conjugate pairs of non-real solutions.

The second product is always positive. If some m_i were odd, then the first product would change sign at a_i, hence $W(x)$ would change sign. Thus all m_i are even. So each factor $(x - a_i)^{m_i}$ in the first product is a square.

Each factor $(x - q_j)(x - \overline{q_j})$ in the second product is a sum of two squares. The product of sums of squares is again a sum of squares, which proves the assertion.

[In fact more is true: since $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$, we can conclude by induction that $W(x)$ is in fact a sum of two squares.]

5. Let S be 6 distinct points in the plane. Let M be the maximum distance between two points of S and let m be the minimum distance between two points of S. Show that $M/m \geq \sqrt{3}$.

Proof. First we show that some three of the six points form an angle greater than 120 degrees. If not all six points are on the boundary of the convex hull of the six points, then some point lies inside a triangle whose vertices are on the boundary of the convex hull. Hence that point and some two of the vertices of the containing triangle form an angle greater than or equal 120 degrees. Otherwise, all six points are on the boundary of the convex hull. But then some consecutive three points on this boundary form an angle greater than or equal to 120 degrees.

Now suppose ABC is any triangle with the angle at C greater or equal 120 degrees. Let a, b, and c be, respectively, the side lengths opposite the vertices A, B, and C. Then, by the law of cosines,

$$c^2 = a^2 + b^2 - 2ab\cos C \geq a^2 + b^2 + ab, \qquad \text{since } \cos C \leq -1/2.$$

Assuming $a \leq b$, we obtain $c^2 \geq 3a^2$, or $c/a \geq \sqrt{3}$. Since $M \geq c$ and $m \leq a$, it follows that $M/m \geq \sqrt{3}$.

6. Let f_1, f_2, f_3 be linearly independent real-valued functions defined on $\mathbb{R}$. Prove that there exist $a_1, a_2, a_3 \in \mathbb{R}$ such that the matrix

$$\begin{bmatrix} f_1(a_1) & f_1(a_2) & f_1(a_3) \\ f_2(a_1) & f_2(a_2) & f_2(a_3) \\ f_3(a_1) & f_3(a_2) & f_3(a_3) \end{bmatrix}$$

is nonsingular. [Recall that functions f_1, f_2, f_3 are said to be *linearly independent* iff the only constants c_1, c_2, c_3 such that $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all x are $c_1 = c_2 = c_3 = 0$.]

Solution I:

Assume (for purposes of contradiction) that for all a_1, a_2, a_3, the matrix is singular. Then the matrix has determinant zero. Write $a_3 = x$ in the determinant, and expand by the third column to get

$$\begin{aligned} 0 &= \begin{vmatrix} f_1(a_1) & f_1(a_2) & f_1(x) \\ f_2(a_1) & f_2(a_2) & f_2(x) \\ f_3(a_1) & f_3(a_2) & f_3(x) \end{vmatrix} \\ &= \begin{vmatrix} f_2(a_1) & f_2(a_2) \\ f_3(a_1) & f_3(a_2) \end{vmatrix} f_1(x) - \begin{vmatrix} f_1(a_1) & f_1(a_2) \\ f_3(a_1) & f_3(a_2) \end{vmatrix} f_2(x) + \begin{vmatrix} f_1(a_1) & f_1(a_2) \\ f_2(a_1) & f_2(a_2) \end{vmatrix} f_3(x). \end{aligned}$$

For any fixed a_1, a_2, this holds for all x. Since f_1, f_2, f_3 are linearly independent, it follows that all three of the 2×2 determinants are 0. Now repeat: Because

$$\begin{vmatrix} f_1(a_1) & f_1(a_2) \\ f_2(a_1) & f_2(a_2) \end{vmatrix} = 0$$

for all a_1, a_2, write $a_2 = x$ and expand to get

$$0 = \begin{vmatrix} f_1(a_1) & f_1(x) \\ f_2(a_1) & f_2(x) \end{vmatrix} = f_1(a_1) f_2(x) - f_2(a_1) f_1(x) + 0 f_3(x)$$

for all x. By linear independence, all these coefficients are 0. So $f_1(a_1) = 0$ for all a_1. Thus f_1 is the zero function, which contradicts linear independence.

Therefore the assumption we started with was wrong: Points a_1, a_2, a_3 do exist so that the 3×3 matrix is nonsingular.

Solution II.

Since the f_i are linearly independent, none are the zero function. Let k be the largest integer such that there exist $a_1, \dots, a_k$ such that the $3 \times k$ matrix $A = (f_i(a_j))$ has rank k. Then $k \geq 1$. By way of contradiction, suppose $k < 3$. Then there exists a nonzero row vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $\mathbf{v}A = 0$. Form the column vector $F(x) = (f_1(x), f_2(x), f_3(x))^t$. Then, since k was chosen maximal, $F(x)$ must be a linear combination of the columns of A. Hence there exists a column vector $\mathbf{w}_x^t$ in $\mathbb{R}^k$ such that $F(x) = A\mathbf{w}_x^t$. Thus $\mathbf{v}F(x) = \mathbf{v}A\mathbf{w}_x^t = 0$. This means that the three given functions are linearly dependent, contradiction. So $k = 3$ and we are done.