Gordon solutions

1. Do there exist 2008 distinct positive integers $n_1, n_2, \cdots, n_{2008}$ such that

$$2 = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_{2008}} \qquad ?$$

I claim, in fact, 2 can be written as a sum of the reciprocals of k distinct positive integers for any $k \geq 4$. Indeed,

$$2 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6}, \qquad 2 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18}$$

and the last term $1/(2N)$ can always be replaced by $1/(3N) + 1/(6N)$.

2. Chameleons on an island come in three colors. They wander and meet in pairs. When two chameleons of different colors meet, they both change to the third color. Given that the initial amounts of the chameleons of the three colors are 13, 15, and 17, show that it may not happen that, after a while, all of them acquire the same color.

Let x be the number of times colors A and B meet, y the number of times colors A and C meet, and z the number of times colors B and C meet. If we end up with all of color A, then we have the system of three linear equations

$$13 - x - y + 2z = 45, \quad 15 - x + 2y - z = 0, \quad 17 + 2x - y - z = 0.$$

This system has no solution all in integers, for example by Gaussian elimination. The same reasoning works for right-hand sides 0,45,0 and for 0,0,45. So it is impossible.

Let a, b, c be the number of chameleons of each color. Note that under any of the three possible changes, the value $b-a$ changes by a multiple of 3. [a and b both decrease by 1 so $b-a$ remains unchanged; or a decreases by 1 and b increases by 2 so $b-a$ increases by 3; or a increases by 2 and b decreases by 1 so $b-a$ decreases by 3.] But $b-a$ begins at 2, so it can never reach 0, it can never reach 45, and it can never reach -45.

Represent the colors by 0, 1, and 2, for, respectively, the 13, 15, and 17 chameleons. Let a and b be the distinct colors of two chameleons who meet and let c be the third color. Then $a+b = -c \pmod 3$. When the two chameleons change color, they both become color c, and $c + c = 2c = -c \pmod 3$; so color changes do not change, modulo 3, the sum of the colors of all chameleons. Since the total number of chameleons is a multiple of 3, if all acquired the same color, then the color sum would be $0 \pmod 3$. However, the initial color sum is $13 \times 0 + 15 \times 1 + 17 \times 2 = 1 \pmod 3$.

3. Let $a_1, a_2, \cdots, a_6$ be positive real numbers with sum 1. Prove:

$$\left(\frac{1}{a_1}-1\right)\left(\frac{1}{a_2}-1\right)\cdots\left(\frac{1}{a_6}-1\right) \geq 2008.$$

Apply the arithmetic-mean/geometric-mean inequality.

$$\left(\frac{1}{a_1}-1\right)\left(\frac{1}{a_2}-1\right)\cdots\left(\frac{1}{a_6}-1\right) = \left(\frac{1-a_1}{a_1}\right)\left(\frac{1-a_2}{a_2}\right)\cdots\left(\frac{1-a_6}{a_6}\right)$$

$$= \frac{(a_2+a_3+a_4+a_5+a_6)(a_1+a_3+a_4+a_5+a_6)\cdots(a_1+a_2+a_3+a_4+a_5)}{a_1a_2a_3a_4a_5a_6}$$

$$\geq \frac{5^6\sqrt[5]{a_2a_3a_4a_5a_6}\sqrt[5]{a_1a_3a_4a_5a_6}\cdots\sqrt[5]{a_1a_2a_3a_4a_5}}{a_1a_2a_3a_4a_5a_6} \qquad \text{[There are 6 radicals, each } a_i \text{ appears 5 times.]}$$

$$= \frac{5^6\left(\sqrt[5]{a_1}\right)^5\left(\sqrt[5]{a_2}\right)^5\cdots\left(\sqrt[5]{a_6}\right)^5}{a_1a_2a_3a_4a_5a_6} = 5^6 > 2008.$$

4. A **lattice point** in the Cartesian plane is a point with both coordinates integers. Three lattice points A, B, C are given. Must there exist three more lattice points A', B', C' such that the triangle $\triangle A'B'C'$ is similar to $\triangle ABC$ but with 5 times the area?

Let ABC be the given triangle labelled clockwise. Erect externally on AB a rectangle $ABB'D$ with $B'D$ twice as long as AB. Then B' is a lattice point and AB' has length $\sqrt{5}$ times the length of AB. Erect internally on AC a rectangle $ACC'E$ with $C'E$ twice as long as AC. Then C' is a lattice point and AC' has length $\sqrt{5}$ times the length of AC. Further, $\angle B'AC' = \angle BAC$. Therefore, $\triangle AB'C'$ is similar to $\triangle ABC$ and has area 5 times larger.

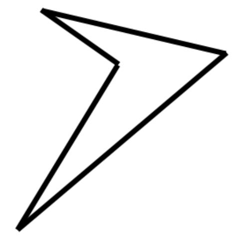
Orthogonal matrices preserve similarity. The orthogonal matrix

$$\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

transforms lattice points to lattice points and has determinant 5.

Using complex numbers: multiply by $2 + i$. Multiplying by a complex number only rotates and scales, thus preserving similarity.

5. A **dart** is a non-convex 4-sided polygon in the plane. Can a convex polygon be partitioned into finitely many darts?



A convex n-gon C has vertices with angles totaling $(n-2)\pi$. Suppose it has been tiled with k darts. Each dart has an angle exceeding π which must lie in the interior of C. Since the large angles prevent two of these vertices being placed at the same point, they comprise k interior points with angles of $2k\pi$ to be covered. Thus, the angles of the darts must cover angles totaling $(2k+n-2)\pi$, but the darts only contain angles totaling $2k\pi$. That is, no tiling is possible.

[Source: Allen J. Schwenk, *Mathematics Magazine*, **60** (1987) p. 182.]

6. Let A and B be $n \times n$ real matrices. Suppose that $I - AB$ has an inverse, where I is the $n \times n$ identity matrix. Show that $I - BA$ also has an inverse.

Deny. Then there exists a nonzero vector v such that $(I - BA)v = 0$. Hence $0 = A0 = A(I - BA)v = (I - AB)Av$. Since $I - AB$ has an inverse, it follows that $Av = 0$. But then $0 = (I - BA)v = v - BAv = v$, contradiction.

Let X be the inverse of $I - AB$. Then

$$\begin{aligned} I &= I - BA + BA = I - BA + BX(I - AB)A \\ &= I - BA + BXA(I - BA) = [I + BXA](I - BA). \end{aligned}$$

so $I + BXA$ is the inverse of $I - BA$.