Solutions to 2013 Gordon Prize examination problems

1. *Prove that the first 2013 digits after the decimal point in the decimal expansion of the number* $(6+\sqrt{37})^{2013}$ *are zero.*

Solution. Let $a=\sqrt{37}+6$ and $b=\sqrt{37}-6=a^{-1}$. Then for odd n, $M_n=a^n-b^n$ is an integer, since

$$M_n=(\sqrt{37}+6)^n-(\sqrt{37}-6)^n=\sum_{k=0}^{n}\tbinom{n}{k}(\sqrt{37})^k6^{n-k}-\sum_{k=0}^{n}(-1)^{n-k}\tbinom{n}{k}(\sqrt{37})^k6^{n-k}$$
$$=\sum_{\substack{0\le k\le n\\ n-k\text{ is odd}}}2\tbinom{n}{k}(\sqrt{37})^k6^{n-k}=2\sum_{\substack{0\le k\le n\\ k\text{ is even}}}\tbinom{n}{k}(\sqrt{37})^k6^{n-k}=2\sum_{\substack{0\le k\le n\\ k\text{ is even}}}\tbinom{n}{k}37^{k/2}6^{n-k}.$$

Now, $b<10^{-1}$ (since $ab=1$ and $a>12$). Hence, for any odd n, $a^n=M_n+b^n$, where M_n is integer and $b^n<10^{-n}$, so (the decimal expansion of) a^n has $\ge n$ zeros after the decimal point.

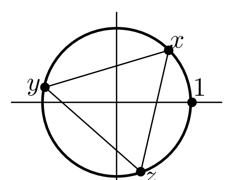
2. *Prove that for any real square matrix* A, $\det(I+A^2)\ge 0$.

Solution. We have $I+A^2=(I+iA)(I-iA)$, and, since A is real, $I-iA=\overline{I+iA}$, the complex conjugate of $I+iA$. So,

$$\det(I+A^2)=\det(I+iA)\cdot\det\overline{(I+iA)}=\det(I+iA)\cdot\overline{\det(I+iA)}=|\det(I+iA)|^2\ge 0.$$

3. *Suppose that real numbers* a, b, c *satisfy the equalities* $\cos a+\cos b+\cos c=\sin a+\sin b+\sin c=0$. *Prove that* $\cos 2a+\cos 2b+\cos 2c=\sin 2a+\sin 2b+\sin 2c=0$.

Solution. Put $x=\cos a+i\sin a$, $y=\cos b+i\sin b$, $z=\cos c+i\sin c$. Then $|x|=|y|=|z|=1$ and $x+y+z=0$. Notice that also $x^{-1}+y^{-1}+z^{-1}=\bar x+\bar y+\bar z=\overline{x+y+z}=0$, so, $xy+yz+zx=0$. Hence, $x^2+y^2+z^2=(x+y+z)^2-2(xy+yz+zx)=0$. Since $x^2=\cos 2a+i\sin 2a$, $y^2=\cos 2b+i\sin 2b$ and $z^2=\cos 2c+i\sin 2c$ we get the result.



Another solution. The equations $|x|=|y|=|z|=1$ and $x+y+z=0$ imply that x, y, and z are located at the vertices of an equilateral triangle inscribed in the unit circle.

(Indeed, let $y'=y/x$ and $z'=z/x$, then $|y'|=|z'|=1$ and $1+y'+z'=0$. Hence, $\operatorname{Im}y'=-\operatorname{Im}z'$, so $\operatorname{Re}y'=\pm\operatorname{Re}z'$, so $\operatorname{Re}y'=\operatorname{Re}z'=-1/2$, and so $y'=e^{2\pi i/3}$ and $z'=e^{4\pi i/3}$, or vice versa.) It follows that the points x^2,y^2,z^2 are also located at the vertices of an equilateral triangle inscribed in the unit circle (now $y^2/x^2=e^{4\pi i/3}$ and $z^2/x^2=e^{2\pi i/3}$, or vice versa), and so, $x^2+y^2+z^2=0$.

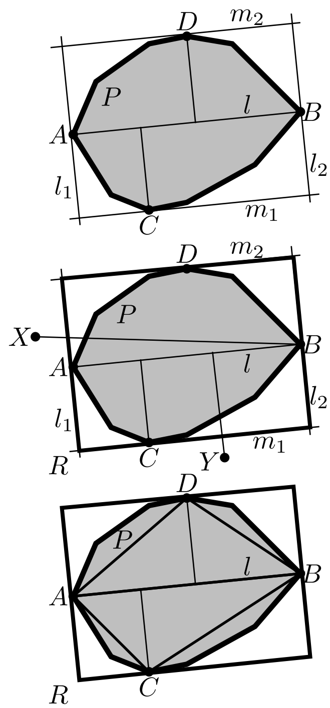
4. *Prove that any positive rational number can be obtained from the number 1 by applying the operations* $x\mapsto x+1$ *and* $x\mapsto\frac{x}{x+1}$.

Solution. The operations $S(x)=x+1$ and $T(x)=\frac{x}{x+1}$ act on the set of positive rational numbers in the folowing way: $S\big(\frac{k}{l}\big)=\frac{k+l}{l}$, $T\big(\frac{k}{l}\big)=\frac{k}{k+l}$. Therefore, to prove that the quotient $\frac{n}{m}$, where $n,m\in\mathbb{N}$ are relatively prime, can be obtained from 1 we need to show that the pair $(1,1)$ can be transformed to the pair (n,m) using the operations $(k,l)\mapsto(k+l,l)$ and $(k,l)\mapsto(k,l+k)$. This is equivalent to showing that the pair (n,m) can be transformed, using a sequence of operations $(k,l)\mapsto(k-l,l)$ if $k>l$ and $(k,l)\mapsto(k,l-k)$ if $l>k$, to the pair $(1,1)$. But this sequence of operations is uniquely defined and, since n and m are relatively prime, always ends with $(1,1)$ (this is just the Euclidean algorithm). (This proves, by the way, that there is only one sequence of the operations T and S that produces the quotient $\frac{n}{m}$.)

Another solution. More formally, for any positive rational number $r=\frac{n}{m}$, written in the lowest terms (that is, with n and m relatively prime), define $C(r)=n+m$, and prove the assertion using the induction on

$C(r)$. The minimal value of $C(r)$ is 2, which is only reached when $r = 1$. Let $r = \frac{n}{m} \in \mathbb{Q}$, where $n \neq m$ and n and m are relatively prime. If $n > m$, put $s = \frac{n-m}{m}$; then $r = s + 1$ and $C(s) = n < C(r)$. If $n < m$, put $s = \frac{n}{m-n}$; then $r = \frac{s}{s+1}$ and $C(s) = m < C(r)$. In both cases, by induction on $C(r)$, s can be obtained from 1 with the prescribed operations, and so can r.

5. *Prove that any convex polygon of area S in the plane is contained in a rectangle of area $2S$.*



Solution. Let P be a convex polygon. Let A and B be the points of P at the maximum distance from each other. Let l_1 and l_2 be the lines orthogonal to the line $l = (AB)$ and passing through A and B respectively. Let C and D be the points of P, in the both half-planes to which l subdivides the plane, whose distance from l is maximal, and let m_1, m_2 be the lines parallel to (AB) passing through the points C and D respectively. Let R be the rectangle bounded by the lines l_1, l_2, m_1, m_2.

We claim that $P \subseteq R$, that is, all points of P lie between l_1 and l_2, and between m_1 and m_2. Indeed, if there is a point X of P not between l_1 and l_2, say, on the other side from l_1, then $\operatorname{dist}(X, B) > \operatorname{dist}(A, B)$, which contradicts the choice of A, B. And if there is a point Y of P not between m_1 and m_2, say, on the other side from m_1, then $\operatorname{dist}(Y, l) > \operatorname{dist}(C, l)$, which contradicts the choice of C.

Next, we claim that $\operatorname{area}(R) \leq 2 \operatorname{area}(P)$. Indeed, since P is convex, the triangles $\triangle ABC$ and $\triangle ABD$ are contained in P, so $\operatorname{area}(P) \geq \operatorname{area}(\triangle ABC) + \operatorname{area}(\triangle ABD)$. On the other hand, $\operatorname{area}(R) = |AB| \cdot \big(\operatorname{dist}(C, l) + \operatorname{dist}(D, l)\big) = 2\big(\operatorname{area}(\triangle ABC) + \operatorname{area}(\triangle ABD)\big)$.

6. *Let $f\colon \mathbb{R} \longrightarrow (0, \infty)$ be a continuous periodic function having period 1; prove that $\int_0^1 \frac{f(x)dx}{f(x+\frac{1}{2013})} \geq 1$.*

Solution. Let n be any positive integer; we will prove that $\int_0^1 \frac{f(x)dx}{f(x+\frac{1}{n})} \geq 1$. Define $g_0(x) = \frac{f(x)}{f(x+\frac{1}{n})}$, and let $g_k(x) = g(x + k/n)$, $k = 1, \ldots, n-1$. Then $\prod_{k=0}^{n-1} g_k(x) = \prod_{k=0}^{n-1} \frac{f(x+\frac{k}{n})}{f(x+\frac{k+1}{n})} = \frac{f(x)}{f(x+1)} = 1$, so, by the generalized arithmetic-geometric mean inequality, $\frac{1}{n}\sum_{k=0}^{n-1} g_k(x) \geq \sqrt[n]{\prod_{k=0}^{n-1} g_k(x)} = 1$. Since, for any k, $\int_0^1 g_k(x)\,dx = \int_0^1 g_0(x)\,dx$, we get

$$\int_0^1 g_0(x)dx = \frac{1}{n}\sum_{k=0}^{n-1}\int_0^1 g_k(x)\,dx = \int_0^1 \Big(\frac{1}{n}\sum_{k=0}^{n-1} g_k(x)\Big)dx \geq \int_0^1 1\,dx = 1.$$

Another solution. We will now prove that $\int_0^1 \frac{f(x)dx}{f(x+a)} \geq 1$ for any $a > 0$ using Jensen's inequality (which says that for any integrable on $[0,1]$ function h and any convex function φ, $\int_0^1 \varphi(h(x))\,dx \geq \varphi\big(\int_0^1 h(x)\,dx\big)$; in particular, $\int_0^1 e^{h(x)}dx \geq e^{\int_0^1 h(x)dx}$).

Take $h(x) = \log(f(x)/f(x+a)) = \log f(x) - \log f(x+a)$; then $e^{h(x)} = \frac{f(x)}{f(x+a)}$, and $\int_0^1 h(x)\,dx = \int_0^1 \log f(x)\,dx - \int_0^1 \log f(x+a)\,dx = 0$, since $\log f(x)$ is periodic with period 1. Hence, by Jensen's inequality, $\int_0^1 \frac{f(x)dx}{f(x+a)} \geq e^0 = 1$.