Solutions: 2002 Rasor-Bareis Prize Examination

1. Recall that binomial coefficient $\binom{k}{r}$ may be written as $k!/(r!(k-r)!)$. The prime factorization of 2002 is $2 \cdot 7 \cdot 11 \cdot 13$. In order for an integer to be divisible by 2002, is is necessary and sufficient that it be divisible by all four of these prime factors. First consider the prime 13: if it divides $k!/(r!(k-r)!)$, then $k \geq 13$. So the first row is at least row 13. Next consider the prime 7 in row 13. If $0 \leq r \leq 13$, then either $r \geq 7$ so that 7 divides $r!$ or $r \leq 6$ so that $(13-r) \geq 7$ and 7 divides $(13-r)!$. So every entry in row 13 has a factor of 7 in the denominator $r!(13-r)!$, so the quotient $13!/(r!(13-r)!)$ is not divisible by 7. And therefore not divisible by 2002. Next consider row 14. Now there are *two* factors of 7 in the numerator 14!. As long as r is not $0, 7, 14$ there is only one factor of 7 in the denominator $r!(14-r)!$. If $4 \leq r \leq 10$, then the primes 11 and 13 have one factor in the numerator and none in the denominator. So we only need to make sure our entry is even: $\binom{14}{5}$ has 11 twos in the numerator and 10 twos in the denominator. So 2002 divides $\binom{14}{5}$, and row 14 is the first row in which this happens. [Actually, $\binom{14}{5} = 2002$.]

2. Assume that the (orthogonal) projection of a body B onto a plane P is a disk D of radius r. Then the projection of B onto any straight line L in P coincides with the projection of D onto L, and so is an interval of length $2r$. (Indeed, we may assume that P is the xy-plane and L is the x-axis; then our statement is evident.)

Now, assume that Q is another plane such that the projection of B onto Q is a disk of radius s. If P and Q are parallel, the projections of B onto P and Q are congruent, so $r = s$. If P and Q are not parallel, let L be their line of intersection $P \cap Q$. Then the projection of B onto L is an interval of length $2r$ and simultaneously an interval of length $2s$. So, again, $r = s$.

3. Let A be the lower left corner, and place it at the origin in the Cartesian plane, with $AB = M$ and $AD = N$. Assume the billiard ball moves with speed v. So the projection of the ball on the x-axis moves from A to B and back with constant speed $v\sqrt{3}/2$, and the projection of the ball on the y-axis moves from A to D and back with constant speed $v/2$. Suppose that the ball returns to the point A at some time T, the x-projection having passed from A to B and back m times, and the y-projection from A to D and back n times. Then one has $2mM = Tv\sqrt{3}/2$ and $2nN = Tv/2$. Dividing the first equation by the second one we get $\sqrt{3} = (mM)/(nN)$, which is impossible since $\sqrt{3}$ is an irrational number. So, in fact, the ball can never return to A.

Problem 4: see the Gordon solutions.

5. *Answer:* No. *Proof.* For an infinite sector S and $R > 0$, let $l(S, R)$ denote the arc length of the intersection of S with the circle of radius R centered at the origin. Then $\lim_{R\to\infty} l(S,R)/R = \alpha$, where α is the angle of S (measured in radians). Let $S_1, \ldots, S_{2002}$ be our given infinite sectors with respective angles $\alpha_1, \ldots, \alpha_{2002}$. Then

$$\lim_{R\to\infty} \frac{1}{R} \sum_{k=1}^{2002} l(S_k, R) = \sum_{k=1}^{2002} \lim_{R\to\infty} \frac{l(S_k, R)}{R} = \sum_{k=1}^{2002} \alpha_k < 2\pi.$$

Hence, there exists $R > 0$ such that $\sum_{k=1}^{2002} l(S_k, R) < 2\pi R$, and therefore the circle of radius R centered at the origin is not contained in $S_1 \cup \ldots \cup S_{2002}$.

6. SOLUTION I

The equilateral triangle with vertices 0, 2002, and $2002u$ contains all the points of L. Its center, $c = (2002/3) + (2002/3)u$ (the average of its three vertices), is not one of the points of L since the imaginary part of c is not an integer multiple of the imaginary part of u. Associate with each point of L its images under the rotations about c through 120 and 240 degrees. Both rotations map L one-to-one onto itself. Thus each point of L is a vertex of an equilateral triangle with center c and all three vertices in L, and L is the disjoint union of such three-element sets. Since L has more than 2/3 of its points colored, at least one of these three-element sets has all of its vertices colored.

SOLUTION II

Let S be the set of all equilateral triangles with vertices $x, x+a, x+au$, where all three are in L and $a > 0$ is an integer. For each triangle from S write its 3 vertices as a column, and place these columns beside each other to form a $3 \times m$ rectangular table T, where m is the number of elements of S. Note that each point from L occurs in T exactly 2002 times, so more than 2/3 of the elements of the table T are colored blue. If each column of T had a non-blue point, at least 1/3 of elements of T would be non-blue, which would be impossible. Hence, there is a column in T whose entries are all blue. The corresponding triangle is an equilateral triangle with blue vertices.