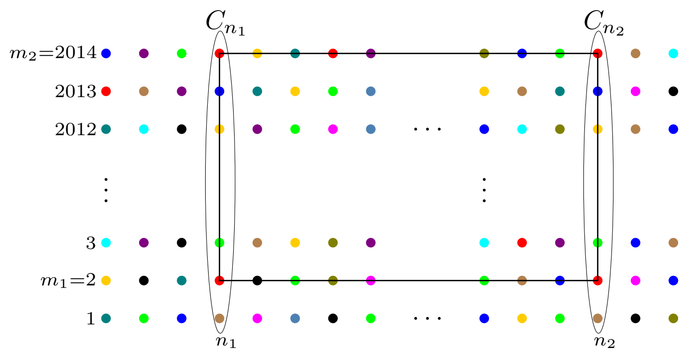
**Solutions to 2013 Rasor-Bareis Prize examination problems**

**1.** *Consider an infinite arithmetical progression of positive integers. Prove that there are infinitely many terms in this progression the sum of whose decimal digits is the same.*

Solution. Let $a$ be the base and $b$ be the step of an arithmetic progression $S$, so that $S$ has the form $(a, a+b, a+2b, \ldots)$. Let the decimal expansion of $a$ be $a_k \ldots a_2 a_1$ and the decimal expansion of $b$ be $b_l \ldots b_2 b_1$. Then the elements $a + 10^k b$, $a + 10^{k+1} b$, $a + 10^{k+2} b$, $\ldots$ of $S$ have expansions $b_l \ldots b_2 b_1 a_k \ldots a_2 a_1$, $b_l \ldots b_2 b_1 0 a_k \ldots a_2 a_1$, $b_l \ldots b_2 b_1 00 a_k \ldots a_2 a_1$, $\ldots$, whose sums of digits are all equal to $\sum_{i=1}^k a_i + \sum_{j=1}^l b_j$.

**2.** *The integer points in the plane are colored with 2013 different colors. Prove that there is a rectangle $\{(n_1,m_1),(n_2,m_1),(n_1,m_2),(n_2,m_2)\}$ whose vertices have the same color.*

Solution. Consider the columns $C_1 = \{1\} \times \{1, \ldots, 2014\}$, $C_2 = \{2\} \times \{1, \ldots, 2014\}$, $C_3 = \{3\} \times \{1, \ldots, 2014\}$, $\ldots$ of height 2014.



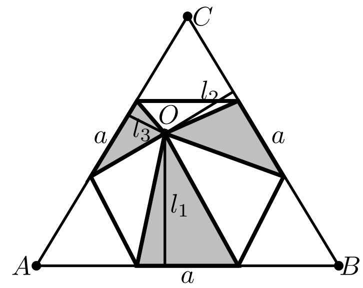

Since there are only finitely many different colorings of a set of 2014 points, among (infinitely many) columns $C_1, C_2, \ldots$ there are two (in fact, infinitely many) having identical colorings; let these be the columns $C_{n_1}$ and $C_{n_2}$. Among 2014 integer points of the column $C_{n_1}$ there are two having the same color; let these be the $m_1$th and the $m_2$th points. Then the points $(n_1,m_1)$, $(n_2,m_1)$, $(n_2,m_2)$, $(n_1,m_2)$ all have the same color.

**3.** *Suppose that $I_1, \ldots, I_n$ are subintervals of $[0,1]$ such that $\sum_{i=1}^n |I_i| \geq 2013$ (where $|I|$ denotes the length of an interval $I$). Prove that there exists a point $x \in [0,1]$ that belongs to at least 2013 of the intervals $I_i$.*

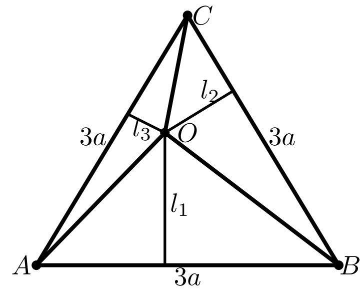
Solution. Let $\{a_0, a_1, \ldots, a_k\}$, with $0 \leq a_0 < a_1 < \ldots < a_k \leq 1$, be the set of the endpoints of the intervals $I_1, \ldots, I_n$, so that each interval $I_i$ has the form $[a_j, a_l]$ for some $j < l$. Let $J_1 = [a_0, a_1]$, $J_2 = [a_1, a_2]$, $\ldots$, $J_k = [a_{k-1}, a_k]$, then $\sum_{j=1}^k |J_j| \leq 1$ and each of $I_i$ is a union of some of the intervals $J_j$. (We assume w.l.o.g. that the intervals $I_i$ are closed.) For each $j = 1, \ldots, k$ let $m_j$ be the number of intervals $I_i$ that contain the subinterval $J_j$; then $\sum_{i=1}^n |I_i| = \sum_{i=1}^n \sum_{j: J_j \subseteq I_i} |J_j| = \sum_{j=1}^k \sum_{i: I_i \supseteq J_j} |J_j| = \sum_{j=1}^k m_j |J_j|$. If $m_j < 2013$ for all $j$, then $\sum_{j=1}^k m_j |J_j| < 2013 \sum_{j=1}^k |J_j| \leq 2013$, which contradicts the assumption; thus $m_j \geq 2013$ for some $j$. This means that every point of the interval $J_j$ is contained in $\geq 2013$ of the intervals $I_i$.

Another solution. For each $i$ let $\chi_i$ be the indicator function of the interval $I_i$, $\chi_i(x) = 1$ if $x \in I_i$ and $\chi_i(x) = 0$ if $x \notin I_i$. Then, for every $i$, $\int_0^1 \chi_i(x)\,dx = |I_i|$. Let $f = \sum_{i=1}^n \chi_i$, then for every $x \in [0,1]$, $f(x)$ is the number of intervals $I_i$ that contain $x$. If $f(x) < 2013$ for all $x$, then $\int_0^1 f(x)\,dx < 2013$; but $\int_0^1 f(x)\,dx = \sum_{i=1}^n \int_0^1 \chi_i(x)\,dx = \sum_{i=1}^n |I_i| \geq 2013$, contradiction. Hence, there exists $x \in [0,1]$ such that $f(x) \geq 2013$.

**4.** *A point inside a regular 6-gon is connected by straight line segments with the vertices, forming six triangles, which are alternately colored black and white. Prove that the sum of the areas of the black triangles is equal to the sum of the areas of the white triangles.*




*Solution*. Call the point inside the 6-gon $O$, and let $a$ be the length of the sides of the 6-gon. Let $B$ be the sum of the areas of the black triangles and $W$ be the sum of the areas of the white triangles. Extend the bases of the black triangles to get an equilateral triangle $\triangle ABC$ with sides of length $3a$. Draw the perpendicular segments from $O$ to the sides of $\triangle ABC$; let the lengths of these segments be $l_1$, $l_2$, and $l_3$. Then $B = \frac{1}{2}a(l_1 + l_2 + l_3)$.




On the other hand, $\frac{1}{2}3a(l_1 + l_2 + l_3)$ is the sum of the areas of the triangles $\triangle AOB$, $\triangle BOC$, and $\triangle COA$, that is, the area of the triangle $\triangle ABC$. Hence, $B = \frac{1}{3}A$, where $A$ is the area of an equilateral triangle with sides of length $3a$. In the same way, $W = \frac{1}{3}A$, so $B = W$.

**5.** *If $x + x^{-1}$ is an integer, prove that $x^{2013} + x^{-2013}$ is also an integer.*

*Solution*. Put $y = x + x^{-1}$. For every integer $n \geq 0$, define $P_n = x^n + x^{-n}$. Then for any $n$, $yP_n = x^{n+1} + x^{-n+1} + x^{n-1} + x^{-n-1} = P_{n-1} + P_{n+1}$. Therefore, the sequence $(P_n)$ satisfies the recurrent equation $P_{n+1} = yP_n - P_{n-1}$ with the initial conditions $P_0 = 2$ and $P_1 = y$. Hence, if $y$ is an integer, $P_n$ are integer for all $n$ by induction on $n$.

Remark. It also follows that for all $n$, $P_n$ is a polynomial function of $y$ with integer coefficients. The polynomial $T_n(y) = P_n(2y)/2$ is called *the $n$-th Chebychev polynomial of the first kind.*

*Another solution*. In the notation above, for any $n \in \mathbb{N}$, using the binomial formula and collecting the symmetric terms we get

$$y^n = (x + x^{-1})^n = x^n + \binom{n}{1}x^{n-1}x^{-1} + \binom{n}{2}x^{n-2}x^{-2} + \ldots + \binom{n}{n-2}x^2x^{2-n} + \binom{n}{n-1}x^1x^{1-n} + x^{-n}$$

$$= (x^n + x^{-n}) + \binom{n}{1}(x^{n-2} + x^{2-n}) + \binom{n}{2}(x^{n-4} + x^{4-n}) + \ldots + \begin{cases} \binom{n}{k} \text{ if } n = 2k \\ \binom{n}{k}(x + x^{-1}) \text{ if } n = 2k+1 \end{cases}$$

$$= P_n + \binom{n}{1}P_{n-2} + \binom{n}{2}P_{n-4} + \ldots + \begin{cases} \binom{n}{k} \text{ if } n = 2k \\ \binom{n}{k}P_1 \text{ if } n = 2k+1. \end{cases}$$

Thus, $P_n$ is a linear combination of $y^n, 1, P_1, \ldots, P_{n-1}$ with integer coefficients, and if $y$ is an integer, then $P_n$ is integer by induction on $n$.

**6.** *Let $f\colon \mathbb{R} \longrightarrow (0, \infty)$ be a continuous periodic function having period 1; prove that $\int_0^1 \frac{f(x)dx}{f(x+1/2)} \geq 1$.*

*Solution*. Let $g(x) = \frac{f(x)}{f(x+1/2)}$. For any $x \in \mathbb{R}$ and $y = x + 1/2$ we have $g(y) = \frac{f(y)}{f(y+1/2)} = \frac{f(x+1/2)}{f(x+1)} = \frac{f(x+1/2)}{f(x)} = \frac{1}{g(x)}$. So,

$$\int_0^1 g(x)\,dx = \frac{1}{2}\Big(\int_0^1 g(x)\,dx + \int_0^1 g(y)\,dy\Big) = \frac{1}{2}\Big(\int_0^1 g(x)\,dx + \int_0^1 \frac{dx}{g(x)}\Big) = \frac{1}{2}\int_0^1 \Big(g(x) + \frac{1}{g(x)}\Big)\,dx$$
$$\geq \frac{1}{2}\int_0^1 2\,dx = 1,$$

since for any $c > 0$ one has $c + \frac{1}{c} \geq 2$.

*Another solution*. We will prove that $\int_0^1 \frac{f(x)dx}{f(x+a)} \geq 1$ for any $a > 0$ using the so-called *Jensen's inequality*, which says that for any integrable on $[0,1]$ function $h$ and any convex function $\varphi$, $\int_0^1 \varphi(h(x))\,dx \geq \varphi\big(\int_0^1 h(x)\,dx\big)$; in particular, $\int_0^1 e^{h(x)}dx \geq e^{\int_0^1 h(x)dx}$.

Take $h(x) = \log(f(x)/f(x+a)) = \log f(x) - \log f(x+a)$; then $e^{h(x)} = \frac{f(x)}{f(x+a)}$, and $\int_0^1 h(x)\,dx = \int_0^1 \log f(x)\,dx - \int_0^1 \log f(x+a)\,dx = 0$, since $\log f(x)$ is periodic with period 1. Hence, by Jensen's inequality, $\int_0^1 \frac{f(x)dx}{f(x+a)} \geq e^0 = 1$.