**Rasor-Bareis solution**

**1.** Prove that, given a rectangle $R$ of area 1, one can place nonoverlapping disks inside $R$ so that the sum of their radii is 2006.

Say the rectangle has dimensions $l \times w$. Since $2006/n$ goes to zero as natural number $n$ increases without bound, there is $n$ so that $a = 2006/n$ is less than the minimum of $l$ and $w$. So there is a square $S$ with side $a$ contained inside $R$. Now divide the square $S$ into $n^2$ small squares by dividing each side into $n$ equal intervals. So we get $n^2$ small squares with side $a/n = 2006/n^2$. Inscribe one disk in each of these small squares, so that it has diameter $2006/n^2$. Then the total of all the diameters is $n^2 \cdot 2006/n^2 = 2006$ exactly.

**Rasor-Bareis solution**

**2.** Let $a$ be a complex number and $n$ a positive integer. Assume $a^n = 1$ and $(a+1)^n = 1$. Show $n$ is a multiple of 6 and $a^3 = 1$.

Since $|a|^n = |a^n| = 1$, the complex number $a$ is on the "unit circle": the circle with center 0 and radius 1. Similarly, $a + 1$ is on the unit circle. But $a + 1$ is one unit to the right of $a$ in the complex plane. So in order for both to be on the unit circle, the line segment joining them must be either the top or bottom side of the regular hexagon inscribed in the unit circle so that one vertex is 1. So $a$ has either argument $2\pi/3$ or $-2\pi/3$. So $a^3 = 1$. And $a+1$ has either argument $\pi/3$ or $-\pi/3$. So in order for the $n$th power of $a+1$ to be 1, we must have $n\pi/3$ a multiple of $2\pi$, so $n$ is a multiple of 6.

## Rasor-Bareis solution

**3.** Let $f$ be a function from reals to reals. Assume that $2f(x) + f(1-x) = x + 4$ for all $x$. Determine the function $f$.

Substitute $1 - x$ for $x$ to obtain $2f(1-x) + f(x) = -x + 5$.
Solve the resulting system of linear equations

$$\begin{cases} 2f(x) + f(1-x) = x + 4, \\ 2f(1-x) + f(x) = -x + 5, \end{cases}$$

for $f(x)$, to obtain $f(x) = x + 1$.

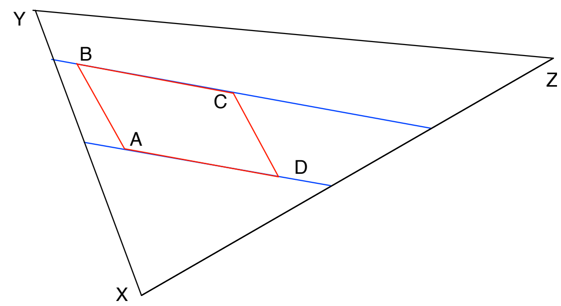
**Rasor-Bareis solution**

**4.** There is an integer $N > 100$ such that $N$ is a square, the last digit of $N$ (in base ten) is not 0, and when the last two digits are deleted, the result is still a square. What is the largest $N$ with this property?

Let $N$ be $a^2$ and let $N$ with the last two digits deleted be $b^2$. So $a$ and $b$ are positive integers with $(10b)^2 < a^2 < (10b)^2 + 100$. But then $10b < a$, so $10b + 1 \leq a$ and $100b^2 + 20b + 1 \leq a^2 < 100b^2 + 100$. From this we get $20b + 1 < 100$, $b < 99/20$, so $b \leq 4$. Then $40^2 = 1600$, $41^2 = 1681$, $42^2 > 1700$. So the largest $N$ is 1681.

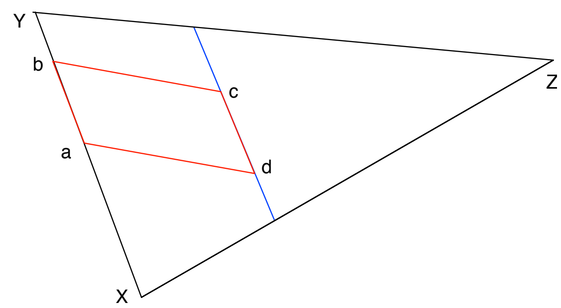
**Rasor-Bareis solution**

**5.** Let $T$ be a triangle in the plane, and let $P$ be a parallelogram that lies inside $T$. Show that the area of $P$ is at most half the area of $T$.

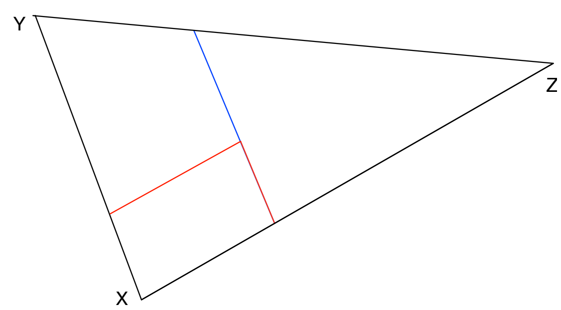



Label the paralellogram $ABCD$ and the triangle $XYZ$.

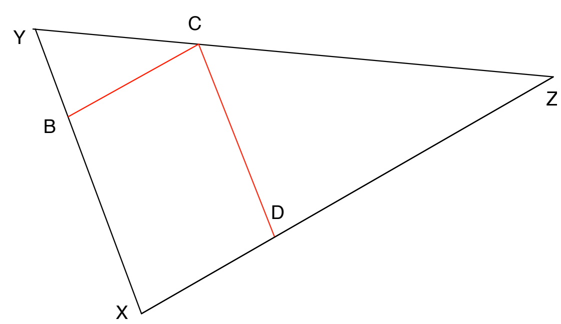
Relabelling, we may assume that the lines $AD$, $BC$ both intersect the same side $XY$ of the triangle. Sliding points $A, D$ along line $AD$ preserving the distance between them preserves the area of the paralellogram. Sliding points $B, C$ along the line $BC$ preserving the distance between them also preserves the area of the paralellogram. Thus we construct a parallelogram $abcd$ of the same area as $ABCD$, with two of the vertices $a, b$ on the interval $XY$.



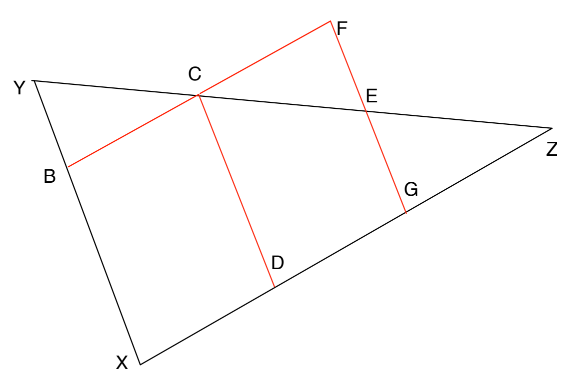

Sliding $a, b$ on $XY$ and $c, d$ on the line $cd$, preserving distances $ab, cd$, we construct a parallelogram of the same area as $ABCD$, with one of the vertices coinciding with vertex $X$ of the triangle, and two other vertices on the sides of the triangle through $X$.




Thus (perhaps increasing the area) we can assume that the the fourth vertex lies on the side $YZ$ of the triangle. Say the paralellogram is $XBCD$ with $B$ on side $XY$, $C$ on side $YZ$ and $D$ on side $ZX$.




Now we may assume $|YC| \leq |ZC|$; if not reverse the labels $Y, Z$ and $B, D$. So there is a point $E$ on $CZ$ with $|YC| = |CE|$. Extend line $BC$ and draw a line through $E$ parallel to $CD$ to get parallelogram $CDFE$ congruent to $XBCD$. But triangle $BYC$ is congruent to triangle $FEC$, so the area of $XYZ$ is double the area of $XBCD$ plus the area of $GEZ$. Thus the area of $XBCD$ is at most half the area of $XYZ$.



**Rasor-Bareis solution**

**6.** Let $f\colon (a,b) \to \mathbb{R}$ be twice continuously differentiable, and assume $f''(x) \neq 0$ for all $x \in (a,b)$. Show that two chords on the graph of $f$ cannot bisect each other. (A **chord** on the graph is a line segment that joins two points on the graph.)

*Proof.* Deny the result. Let $AB$ and $CD$ be the two chords of the graph of $f$ which bisect each other. Then $ACBD$ is a parallelogram. Let $x_A, x_B, x_C$, and $x_D$ be the $x$-coordinates, resp., of $A, B, C$, and $D$. Without loss we may assume that $x_A < x_C < x_B < x_D$. Hence the chords $AC$ and $BD$ are parallel in disjoint intervals $(x_A, x_C)$ and $(x_B, x_D)$. By the mean value theorem $f'(x)$ has the same value for an element in both intervals $(x_A, x_C)$ and $(x_B, x_D)$. But then $f''(x) = 0$ for some $x$ in $(a,b)$, contradiction.

**Gordon solution**

**1.** There is an integer $N > 100$ such that $N$ is a square, the last digit of $N$ (in base ten) is not 0, and when the last two digits are deleted, the result is still a square. What is the largest $N$ with this property?

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**Gordon solution**

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**Gordon solution**

**3.** Let $A = (a_{ij})$ be a $2006 \times 2006$ "checkerboard" matrix of 0s and 1s. That is, $a_{ij} = 0$ if $i + j$ is even and $a_{ij} = 1$ if $i + j$ is odd. Compute the characteristic polynomial of $A$.

Since the odd columns of $A$ are all equal and the even columns of $A$ are all equal, the 2004 vectors $(0, \cdots, 1, 0, -1, 0, \cdots, 0)$, with the 1 of the triple $1, 0, -1$ ranging from the first position to the 2004th position, are linearly independent eigenvectors for the eigenvalue 0. Since all row sums are equal to 1003, the vector $(1, \cdots, 1)$ of all 1s is an eigenvector of $A$ for the eigenvalue 1003. Finally, the vector $(1, -1, 1, -1, \cdots, 1, -1)$ of alternating 1s and $(-1)$s is an eigenvector for the eigenvalue $-1003$. Hence the characteristic polynomial of $A$ is $x^{2004}(x^2 - 1003^2)$.

*Remark.* As an alternate method to find the zero eigenvalues, note that there are only two different rows, so the row space has dimension 2, so the matrix has rank 2. Therefore, all but 2 of the eigenvalues are zero.

**Gordon solution**

**4.** A sequence $\{a_n\}$ of **positive** real numbers satisfies $a_0 = 1$ and $a_{n+2} = 2a_n - a_{n+1}$ for $n \geq 0$. (Note that $a_1$ is not specified.) Find $a_{2006}$; justify your answer.

Answer: $a_{2006} = 1$.

*Proof I.* Note that by the recursion formula,

$$\begin{aligned} a_2 &= -1(a_1) + 2 \\ a_3 &= 3(a_1) - 2 \\ a_4 &= -5(a_1) + 6 \\ a_5 &= 11(a_1) - 10 \\ &\vdots \end{aligned}$$

which suggests the following pattern: for every $n$,there exists an integer $m_n$ so that $a_n = m_n(a_1) + (-m_n + 1)$. This is easily proved by induction on $n$: we already see that the hypothesis is true for $n = 2$ and $n = 3$. Suppose now that it is true for all $n$ less than some $N > 3$. Then, $a_N = 2a_{N-2} - a_{N-1} = 2(m_{N-2}(a_1) + (-m_{N-2}+1)) - (m_{N-1}(a_1) + (-m_{N-1}+1)(a_1)) = (2m_{N-2} - m_{N-1})(a_1) + (-(2m_{N-2} - m_{N-1}) + 1)$, proving the assertion for $n = N$ with $m_N = 2m_{N-2} - m_{N-1}$.

We also claim that for $n > 1$, $m_n$ is negative for even $n$ and positive for odd $n$. This is again proved by induction and the recursive formula for $m_n$ which was just discovered. This implies that for every $n > 3$, $m_{n-2}$ and $m_{n-1}$ have opposite signs, and therefore that $m_n = 2m_{n-2} - m_{n-1}$ implies $|m_n| > |m_{n-1}|$. This means that $\lim_{n\to\infty} |m_n| = \infty$.

Now, suppose that $a_1 \neq 1$. If $a_1 > 1$, then there exists a positive integer $L$ so that $a_1 > 1 + \frac{1}{L}$. Since $\lim_{n\to\infty} |m_n| = \infty$, there exists an even $n$ so that $|m_n| > L$. Then, since all elements of $\{a_n\}$ are positive, $a_n = m_n(a_1) + (-m_n+1) = -|m_n|(a_1) + (|m_n| + 1) > 0$. Therefore, $a_1 < \frac{|m_n|+1}{|m_n|} = 1 + \frac{1}{|m_n|}$, which is a contradiction since $|m_n| > L$. If $a_1 < 1$, then there exists a positive integer $M$ so that $a_1 > 1 - \frac{1}{M}$. Since $\lim_{n\to\infty} |m_n| = \infty$, there exists an odd $n$ so that $|m_n| > M$. Then, since all elements of $\{a_n\}$ are positive, $a_n = m_n(a_1) + (-m_n+1) = |m_n|(a_1) + (-|m_n| + 1) > 0$. Therefore, $a_1 > \frac{|m_n|-1}{|m_n|} = 1 - \frac{1}{|m_n|}$, a contradiction since $|m_n| > M$. We have then shown that the only possibility is that $a_1 = 1$. The recursion formula $a_{n+2} = 2a_n - a_{n+1}$ then implies that $a_n = 1$ for all $n$, and in particular that $a_{2006} = 1$.

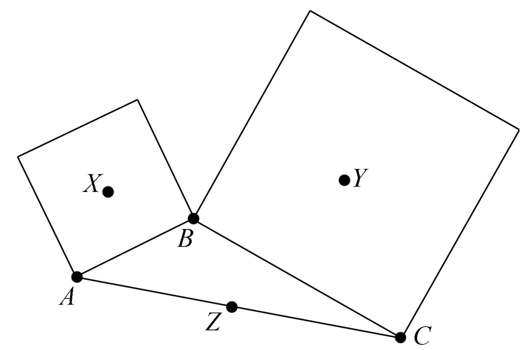
*Proof II.* This is a linear homogeneous difference equation with constant coefficients. The monic polynomial associated with it is $x^2 + x - 2$, which has roots 1 and $-2$. So the solutions of the difference equation all have the form

$$a_n = b(1)^n + c(-2)^n, \quad \text{for } n \geq 0.$$

Now, if $c \neq 0$ then the sequence $a_n$ is eventually alternating, which contradicts the assumption that $a_n$ are all positive. And $a_0 = 1$, so $b = 1$. Therefore $a_n = 1$ for all $n$.

**Gordon solution**

**5.** Let $ABC$ be a triangle in the plane. Erect squares externally on its sides $AB$ and $BC$. Let $X$ and $Y$ be the centers of these squares and let $Z$ be the midpoint of $CA$. Prove that the triangle $XYZ$ is an isosceles right triangle. (It may help to use complex numbers.)




*Proof.* Let $ABC$ be a triangle, labelled clockwise, and let $Z$ be the midpoint of $AC$. The center of the square erected externally on $AB$ is

$$X = A + \frac{B-A}{2} + i\,\frac{B-A}{2} = \frac{(1-i)A + (1+i)B}{2}.$$

The center of the square erected externally on $BC$ is

$$Y = B + \frac{(C-B)}{2} + i\,\frac{C-B}{2} = \frac{(1-i)B + (1+i)C}{2}.$$

Since $Z = (A+C)/2$, we have

$$Y - Z = \frac{(1-i)B + (1+i)C}{2} - \frac{A+C}{2} = \frac{-A + (1-i)B + iC}{2}$$

and

$$X - Z = \frac{(1-i)A + (1+i)B}{2} - \frac{A+C}{2} = \frac{-iA + (1+i)B - C}{2} = i\,(Y - Z).$$

Therefore $X - Z$ and $Y - Z$ are orthogonal and of the same length, so $XYZ$ is an isosceles right triangle.

**Gordon solution**

**6.** For each integer $k > 1$, let $r_k$ be the remainder when $2^{1003}$ is divided by $k$. Prove that $r_2 + r_3 + \cdots + r_{1003} > 2006$.

There are 501 odd integers $k$ with $3 \leq k \leq 1003$, and for each of them we have $r_k \geq 1$, so they give us a total contribution at least 501. There are 250 integers of the form $2(2s+1)$, and they each have remainder at least 2, so their total contribution at least $2 \cdot 250 = 500$. There are 124 integers of the form $4(2s+1)$, their total contribution at least $4 \cdot 124 = 496$. There are 62 integers of the form $8(2s+1)$, their total contribution at least $8 \cdot 62 = 496$. And $k = 48 = 16 \cdot 3$ contributes at least 16. So the total is at least $2009 > 2006$.