Math 35300: Section 001. Linear algebra II **Fall 2012**

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Homework 8

Exercises 1–10 should be regarded as warm-up exercises. They are intended to test your understanding of some of the definitions and constructions introduced in lecture. Your first step to answering these should be to go back to the lecture notes and read again the appropriate definition or construction.

Exercise 1. An inner product on a real vector space V is a map

(a) $\langle\ ,\ \rangle\colon V \times V \longrightarrow \mathbb{R}$
(b) $\langle\ ,\ \rangle\colon V \times V \longrightarrow V$
(c) $\langle\ ,\ \rangle\colon \mathbb{R} \times V \longrightarrow V$

Exercise 2. Positive definiteness of the inner product means that

(a) $\langle x, y\rangle > 0 \Longrightarrow x = y$.
(b) $\langle x, x\rangle > 0 \Longrightarrow x \neq 0$.
(c) $\langle x, x\rangle > 0$ for all $x \in V$, $x \neq 0$.

Exercise 3. Which of the following statements is (or are) correct?

(a) If $\langle\ ,\ \rangle\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is an inner product on the real vector space $\mathbb{R}^n$, then $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$ for all $x, y \in \mathbb{R}^n$.
(b) If $\langle\ ,\ \rangle\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is an inner product on the real vector space $\mathbb{R}^n$, then $\langle x, y\rangle = (x_1y_1, \ldots, x_ny_n)$ for all $x, y \in \mathbb{R}^n$.
(c) If one defines $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$ for all $x, y \in \mathbb{R}^n$, then one obtains an inner product on $\mathbb{R}^n$.

Exercise 4. By the orthogonal complement $U^\perp$ of a subspace U of the Euclidean vector space V, one understands

(a) $U^\perp := \{u \in U \mid u \perp U\}$.
(b) $U^\perp := \{x \in V \mid x \perp U\}$.
(c) $U^\perp := \{x \in V \mid x \perp U \text{ and } \|x\| = 1\}$.

Exercise 5. Let $V = \mathbb{R}^2$ with the standard inner product. Which of the following tuples of elements of V forms an orthonormal basis?

(a) $((1, -1), (-1, -1))$
(b) $((-1, 0), (0, -1))$
(c) $((1, 0), (0, 1), (1, 1))$

Exercise 6. Which of the following conditions on a linear map $f\colon V \longrightarrow W$ of one Euclidean space into another is equivalent to f being orthogonal?

(a) $\langle f(x), f(y)\rangle > 0$ for all $x, y \in V$.
(b) $\langle x, y\rangle = 0 \Longleftrightarrow \langle f(x), f(y)\rangle = 0$.
(c) $\|f(x)\| = \|x\|$ for all $x \in V$.

Exercise 7. For which subspaces $U \subset V$ is the orthogonal projection $P_U\colon V \longrightarrow U$ an orthogonal map?

(a) for each U (b) only for $U = V$ (c) only for $V = \{0\}$

Exercise 8. Which of the following matrices is (or are) orthogonal?

$$\text{(a)}\quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{(b)}\quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{(c)}\quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Exercise 9. Which of the following arguments correctly explains why $(\mathbb{N}, +)$ fails to be a group?

(a) For natural numbers we have $n + m = m + n$, but this is not one of the group axioms, so $(\mathbb{N}, +)$ fails to be a group.
(b) The operation $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$, $(n, m) \longmapsto n + m$, is not defined for all integers, because the negative numbers do not belong to $\mathbb{N}$. Therefore, $(\mathbb{N}, +)$ fails to be a group.
(c) The third group axiom (existence of inverses) is not satisfied, since, for example, for $1 \in \mathbb{N}$ there exists no $n \in \mathbb{N}$ with $1 + n = 0$. Therefore, $(\mathbb{N}, +)$ fails to be a group.

Exercise 10. For $k > 0$ we have

(a) $SO(2k) \subset O(k)$.
(b) $SO(2k) \subset O(2k)$, but $SO(2k) \neq O(2k)$.
(c) $SO(2k) = O(2k)$, because $(-1)^{2k} = 1$.

Exercise 11. Calculate the determinant of the $n \times n$ matrix

$$\begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$$

The expansion by minors formula for determinants, say, on the first column, gives the induction step.

Exercise 12. Let $\mathrm{M}(n \times n, \mathbb{Z})$ denote the set of $n \times n$ matrices with integral coefficients. Let $A \in \mathrm{M}(n \times n, \mathbb{Z})$. Show that there exists $B \in \mathrm{M}(n \times n, \mathbb{Z})$ with $AB = I$ if and only if $\det A = \pm 1$.

Exercise 13. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with determinant 1. What is A^{-1}?

Exercise 14. Prove that expansion by minors on a row of a matrix defines the determinant function.

Exercise 15. Write the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ as a product of elementary matrices, using as few as you can. Prove that your expression is as short as possible.

Exercise 16.

(a) Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b - a)(c - a)(c - b)$.
(b) Prove an analogous formula for such $n \times n$ matrices by using row operations to clear out the first column cleverly.

Exercise 17. Apply the Gram-Schmidt procedure to the basis $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ of $\mathbb{R}^3$, where $\mathbb{R}^3$ is given the usual inner product.

Exercise 18. Prove Pythagoras' theorem: if the three points a, b, c in a Euclidean vector space form a right-angled triangle, that is if $(a-c) \perp (b-c)$, then

$$\|a-c\|^2 + \|b-c\|^2 = \|a-b\|^2$$

Exercise 19. Give $\mathbb{R}^3$ the inner product $\langle x, y\rangle := \sum_{i,j=1}^{3} a_{ij} x_i y_j$, where

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

(That this is actually an inner product is not part of the exercise, and may be assumed). Calculate the cosines of the angles between the canonical unit vectors in $\mathbb{R}^3$.

Exercise 20. Show that the 2×2 matrices $A \in O(2)$, whose coefficients take only the values 0, 1, and -1, form a nonabelian group.

Exercise 21. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $n \geq 2$, define $|x| := \max_i |x_i|$. Show that there exists no inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$, for which $\langle x, x\rangle = |x|^2$ for all $x \in \mathbb{R}^n$.

Exercise 22. Let V be the real vector space of all bounded real sequences,

$$V := \{(x_i)_{i=1,2,\dots} \mid x_i \in \mathbb{R} \text{ and there exists } c \in \mathbb{R} \text{ with } |x_i| \leq c \text{ for all } i\}.$$

Then

$$\langle x, y\rangle := \sum_{n=1}^{\infty} \frac{x_n y_n}{n^2}$$

obviously defines an inner product on V. Find a *proper* vector subspace $U \subset V$ (i.e., $U \neq V$) with $U^\perp = \{0\}$.

Exercise 23. Let M be a set. Show that if $(\mathrm{Bij}(M), \circ)$ is abelian, then M has fewer than three elements.