OSU Analysis Qualifying Examination 1 **August 2018**

Answer each question on a separate sheet or sheets of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Solutions to five problems constitute a complete exam.

Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

This is a two hour, closed book, closed notes exam.

1. Let $(X, \mathcal{A}, \mu)$ be a semifinite measure space and let $f, g : X \to [0, \infty]$ be measurable. Suppose $\int_A f\, d\mu \leq \int_A g\, d\mu$ for each $A \in \mathcal{A}$. Prove that $f \leq g$ μ-a.e. (Note: A proof that does not involve subtraction is to be preferred, since we want avoid subtracting infinity from infinity.)

2. Let $\{f_n\}_{n \in N}$ be a sequence of measurable functions on $[0, 1]$, such that

   (i) $\int_0^1 |f_n(x)|^2\, dx \leq 2018$, $\forall n$, and

   (ii) $f_n \underset{n\to\infty}{\longrightarrow} 0$ a.e.

   Prove that, for any function $g \in L^2([0, 1])$, $\int_0^1 f_n g\, dx \to 0$.

3. Let (X, ρ) be a separable metric space, let $\mathcal{B}$ be the Borel σ-algebra on X, let μ be a finite measure on $\mathcal{B}$, and let $\varepsilon > 0$. Prove that there is a closed totally bounded set $A \subseteq X$ such that $\mu(X \backslash A) < \varepsilon$.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f : X \to [0, \infty)$ be measurable, let $Z = \{f > 1\}$, and suppose $\int_Z f\, d\mu < \infty$. Find $\lim_{n\to\infty} \int_X f^{1/n}\, d\mu$. Justify your answer.

5. Let $f, g : [a, b] \to \mathbb{C}$ be absolutely continuous. Prove that

$$\int_a^b f'(x)g(x)\, dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\, dx\, .$$

   Be sure to explain where and how absolute continuity is used. If your proof contains a step of the form $\int(u + v) = \int u + \int v$, be sure to explain why it is justified. (For example, if $v = -u$ but u is not integrable, then such a step is not justified.)

6. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and suppose $\lim_{n\to\infty} f_n = f_0$ in measure, with $f_n \neq 0$ almost everywhere for all n and $f_0 \neq 0$ almost everywhere. Then show that

$$\lim_{n\to\infty} \frac{1}{f_n} = \frac{1}{f_0} \quad \text{in measure}\, .$$

7. Let $(X, \mathcal{A}, \mu)$ be a measure space with μ a semifinite measure. Suppose E is a measurable set with $\mu(E) = \infty$. Show that for any $C > 0$ there exists a measurable set F, with $F \subset E$, such that $C < \mu(F) < \infty$.