Poncelet's porisms

§0. Porism = a proposition affirming the possibility of finding such conditions as will render a problem indeterminate, or capable of innumerable solutions.

Jean Victor Poncelet (1/7/1788 – 22/12/1867 ; Metz France):

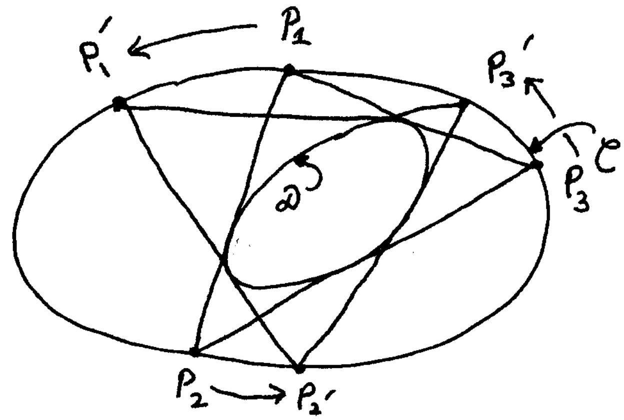
Let $\mathcal{C}$ and $\mathcal{D}$ be two ellipses in $\mathbb{R}^2$; $\mathcal{D}$ within $\mathcal{C}$ (see figure below).

$P_1, P_2, \ldots, P_n \in \mathcal{C}$ s.t. $\overleftrightarrow{P_i P_{i+1}}$ is tangent to $\mathcal{D}$ $\forall\ i = 1, 2, \ldots, n$

distinct points ($P_{n+1} = P_1$)

will be called a Poncelet n-gon about $\mathcal{C}, \mathcal{D}$.

(Cayley used in-and-circumscribed n-gon (about $\mathcal{C}, \mathcal{D}$)).



Theorem. If a Poncelet n-gon exists, about $\mathcal{C}, \mathcal{D}$, then every ~~ver~~ point of $\mathcal{C}$ is a vertex of (its) Poncelet n-gon

$P_1 P_2 P_3$ = Poncelet triangle.

(Traités des propriétés ~~des~~ projective des figures (1822))

- Poncelet proved this theorem during his captivity ((1812–1814) Saratov, Russia, as a prisoner of war (Nepoleon's Russian invasion).

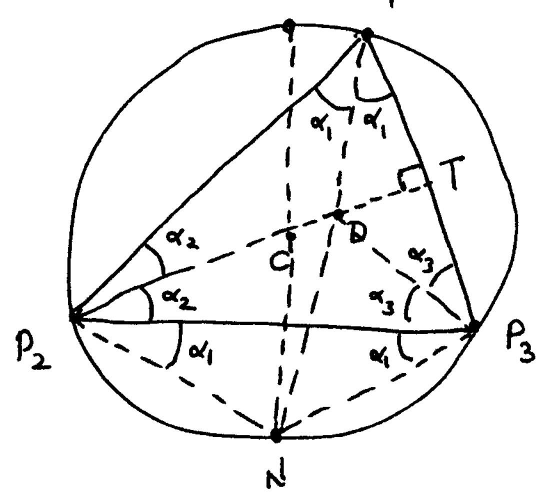
§1. Special case. $\mathcal{C}$ = circle centred at C, of radius R.

$\mathcal{D}$ = " " " D, " " r.

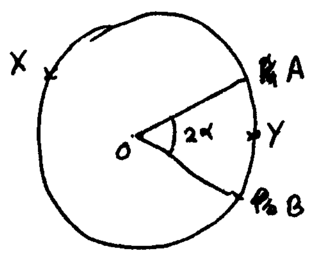
$(a = |CD|)$. Assumption : $r + a < R$.

Euler-Chapple formula. $(n=3)$ Poncelet triangles exist (at any $P \in \mathcal{C}$).

$$\iff \quad r = \frac{(R-a)(R+a)}{2R}.$$



Proof. Use the following fact:



$\Rightarrow \angle AXB = \alpha$

$\angle AYB = \pi - \alpha$

– (*)

Construction. (i) Join P_1, D and let $N := \overleftrightarrow{P_1 D} \cap \mathcal{C}$

(ii) Join N, C and let $M := \overleftrightarrow{NC} \cap \mathcal{C}$

(iii) T = point on $\overline{P_1 P_3}$ s.t. $DT \perp P_1 P_3$

Claim. –
- $\Delta MNP_3 \sim \Delta P_1 DT$
- $|ND| = |NP_2| = |NP_3|$

(pf : $\angle MP_3N = \frac{\pi}{2} = \angle P_1 T D$; $\angle NMP_3 = \angle NP_1P_3 = \angle DP_1T$.

$\angle NP_3D = \alpha_1 + \alpha_3 = \angle NDP_3$ (use (*) for getting these angles).)

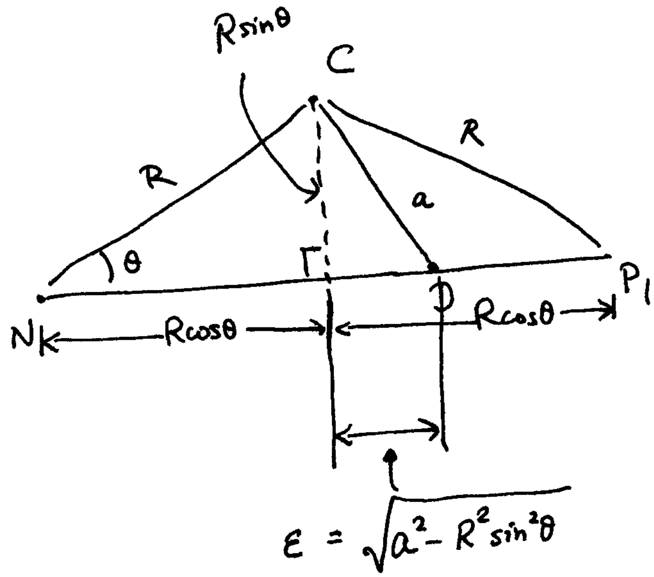
Now. $$\frac{|NP_3|}{|MN|} = \frac{|DT|}{|P_1D|}$$

$$\Rightarrow \quad |ND|\cdot|DP_1| = |DT|\cdot|MN| = r\cdot 2R$$

But $|ND|\cdot|DP_1| =$

$$R^2\cos^2\theta - \varepsilon^2$$

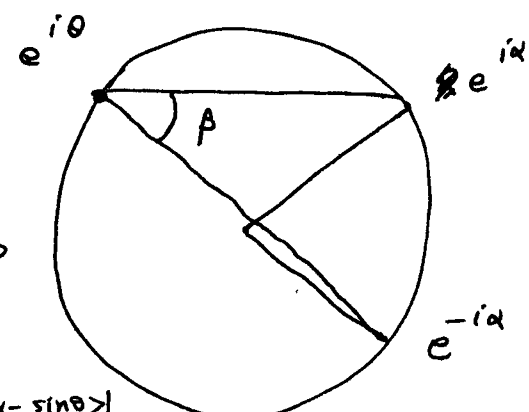
$$= R^2 - a^2.$$



Hence, we get

$$(R-a)(R+a) = (2R)\cdot r.$$

□



Proof of (*) on page (2).

$$\cos(\beta) = \frac{\langle\cos\alpha - \cos\theta, \sin\alpha - \sin\theta\rangle \cdot \langle\cos\alpha - \cos\theta, -\sin\alpha - \sin\theta\rangle}{|\langle\cos\alpha - \cos\theta, \sin\alpha - \sin\theta\rangle|\cdot|\langle\cos\alpha - \cos\theta, -\sin\alpha - \sin\theta\rangle|}$$

$$= \frac{\cos^2\alpha + \cos^2\theta - 2\cos\alpha\cos\theta - \sin^2\alpha + \sin^2\theta}{\sqrt{(2 - 2\cos(\theta+\alpha))(2 - 2\cos(\theta-\alpha))}}$$

$$= \frac{\cancel{2}\cos(\alpha)(\cancel{\cos\alpha} - \cos\theta)}{\cancel{2}\sqrt{(1-\cos(\theta+\alpha))(1-\cos(\theta-\alpha))}} \quad \leftarrow \quad \begin{aligned}&(1-\cos(a-b))(1-\cos(a+b)) \\ &= (\cos(a) - \cos(b))^2\end{aligned}$$

□

§2. Analytic approach

Abel (5/8/1802 - 6/4/1829 ; Nedstrand Norway)

Jacobi (10/12/1804 - 18/2/1851 ; Postdam)

$\mathcal{C}$ = circle of radius R centered at 0 = $\{z \in \mathbb{C} : |z| = R\} \subset \mathbb{C}$

$\mathcal{D}$ = " " " r " " $-a$ = $\{z \in \mathbb{C} : |z+a| = r\} \subset \mathbb{C}$

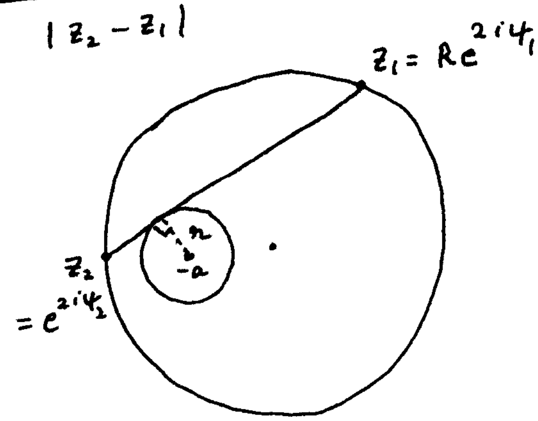
Problem: Given $z_1 = Re^{2i\psi_1}$ and $z_2 = Re^{2i\psi_2}$ on $\mathcal{C}$; determine whether $\overline{z_1 z_2}$ is tangent to $\mathcal{D}$.

Solution : Use the formula: $z_1, z_2, \alpha \in \mathbb{C}$

$$\Rightarrow \text{distance}(\alpha; \overleftrightarrow{z_1 z_2}) = \frac{|\operatorname{Im}((\overline{z_1 - \alpha})(z_2 - z_1))|}{|z_2 - z_1|}$$

with $\alpha = -a$, to get

$$\overline{z_1 z_2} \text{ is tangent to } \mathcal{D} \iff r = \frac{|\operatorname{Im}((\overline{z}_1 + a)(z_2 - z_1))|}{|z_2 - z_1|}.$$



Now $|z_2 - z_1| = 2R\sin(\psi_2 - \psi_1)$

$$\operatorname{Im}[(\overline{z}_1 + a)(z_2 - z_1)]$$
$$= R^2 \sin(2(\psi_2 - \psi_1)) + aR(\sin(2\psi_2) - \sin(2\psi_1))$$

$(\psi_2 - \psi_1 \in (0, \frac{\pi}{2}))$

Hence $$r = R^2 \frac{\sin(2(\psi_2 - \psi_1))}{2R\sin(\psi_2 - \psi_1)} + \frac{aR}{2R^2}\left(\frac{\sin(2\psi_2) - \sin(2\psi_1)}{\sin(\psi_2 - \psi_1)}\right)$$

$$= R\cos(\psi_2 - \psi_1) + a\cos(\psi_2 + \psi_1).$$ We get

$$\boxed{(R+a)\cos\psi_1 \cos\psi_2 + (R-a)\sin\psi_1 \sin\psi_2 = r}$$

□

§3. Explicit solution to the implicitly defined ψ_2.

Restating the computation of §2 above, given $R, r, a \in \mathbb{R}_{>0}$ s.t $(r + a < R)$

We view the following as an equation definining ψ_2 in terms of ψ_1.

$$(R+a)\cos\psi_1 \cos\psi_2 + (R-a)\sin\psi_1 \sin\psi_2 = r. \quad -(1)$$

e.g. $\psi_1 = 0 \implies \cos\psi_2 = \dfrac{r}{R+a}$.

Problem. Compute $\dfrac{d\psi_2}{d\psi_1}$.

Soln. -
$$(R+a)\left(-\sin\psi_1 \cos\psi_2 - \cos\psi_1 \sin\psi_2 \frac{d\psi_2}{d\psi_1}\right)$$
$$+ (R-a)\left(\cos\psi_1 \sin\psi_2 + \sin\psi_1 \cos\psi_2 \frac{d\psi_2}{d\psi_1}\right) = 0$$

$$\implies \frac{d\psi_2}{d\psi_1} = + \frac{(R-a)\cos\psi_1 \sin\psi_2 - (R+a)\sin\psi_1\cos\psi_2}{(R+a)\cos\psi_1 \sin\psi_2 - (R-a)\sin\psi_1 \cos\psi_2} \quad - (2)$$

$\square$

$$\implies \left(\frac{d\psi_2}{d\psi_1}\right)^2 = \frac{(R-a)^2(1-\sin^2\psi_1)\sin^2\psi_2 + (R+a)^2(1-\cos^2\psi_1)\cos^2\psi_2 - 2(R-a)(R+a)\sin\psi_1\cos\psi_1\sin\psi_2\cos\psi_2}{\text{Num w/ } \psi_1 \leftrightarrow \psi_2}$$

by (1)

$$= \frac{(R+a)^2\cos^2\psi_2 + (R-a)^2\sin^2\psi_2 - r^2}{(R+a)^2\cos^2\psi_1 + (R-a)^2\sin^2\psi_1 - r^2}$$

$$= \frac{1 - k^2\sin^2\psi_2}{1 - k^2\sin^2\psi_1} \qquad \text{where } k^2 = \frac{4aR}{(R+a)^2 - r^2}$$

$\in (0,1)$ because $R - a \geq r$

[Separation of variables.]

Hence $$\frac{d\psi_2}{d\psi_1} = \sqrt{\frac{1-k^2\sin^2\psi_2}{1-k^2\sin^2\psi_1}} \quad \text{where } k^2 = \frac{4aR}{(R+a)^2 - r^2}$$

$$\boxed{\frac{d\psi_2}{\sqrt{1-k^2\sin^2\psi_2}} = \frac{d\psi_1}{\sqrt{1-k^2\sin^2\psi_1}}} .$$

§4. Poncelet's Theorem for circles.

Define $\mathrm{am}(u) : \mathbb{R} \to \mathbb{R}$ by $\begin{cases} \mathrm{am}(0) = 0 \\ \dfrac{d}{du}\,\mathrm{am}(u) = \sqrt{1-k^2\sin^2(\mathrm{am}(u))} \end{cases}$.

[always increasing smooth (C^∞)]

By the calculation of §3, $\psi_1 = \mathrm{am}(u) \Rightarrow \psi_2 = \mathrm{am}(u + c)$

where $\mathrm{am}(c) = \cos^{-1}\left(\frac{r}{R+a}\right)$.

Therefore, $\forall\, n \geq 1$, $\psi_n = \mathrm{am}(u + nc)$.

Existence of Poncelet n-gon $\iff \psi_n = \psi_1 + l\pi$ for some $l \in \mathbb{Z}_{\geq 1}$

$\iff \mathrm{am}(u + nc) = \mathrm{am}(u) + l\pi$

Lemma. - $K := \displaystyle\int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}$. Then $\mathrm{am}(u + 2K) = \mathrm{am}(u) + \pi$

Continuing: Poncelet n-gon exists (from $\psi_1 = \mathrm{am}(u)$)

$$\iff \quad \mathrm{am}(u + nc) = \mathrm{am}(u + 2K\ell) \quad . \text{ (for some } \ell \in \mathbb{Z}_{\geq 1})$$

$$\iff \quad n \cdot c = 2K\ell$$

$$\boxed{\frac{c}{K} = \frac{2\ell}{n}}$$ ← no mention of ψ_1.

□

am$(u;k)$ depends on k^2.

§5. Amplitude function – obvious properties.

(i) $\mathrm{am}(-u) = -\mathrm{am}(u)$

(ii) $\mathrm{am}(u + 2K) = \mathrm{am}(u) + \pi$ where

$$K = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$

(iii) $\mathrm{am}(u) \longrightarrow u$ as $k^2 \to 0^+$.

- Defined by Jacobi to compute the arc length of an ellipse.

$$\varphi = \mathrm{am}(u;k) \quad \text{means} \quad u = \int_0^{\varphi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$

$$u = \varphi + O(k^2)$$

$$\left.\begin{aligned} &\bullet \quad \sin(\mathrm{am}(u)) =: \mathrm{sn}(u) \\ &\bullet \quad \cos(\mathrm{am}(u)) =: \mathrm{cn}(u) \\ &\bullet \quad \frac{d}{du}\mathrm{am}(u) =: \mathrm{dn}(u) \end{aligned}\right\} \text{ Jacobi's elliptic functions}$$

§6. Properties on sn ; cn ; dn

$$\mathrm{sn}^2(u) + \mathrm{cn}^2(u) = 1 \qquad \frac{d}{du}\,\mathrm{sn}(u) = \mathrm{cn}(u)\,\mathrm{dn}(u)$$

$$k^2\,\mathrm{sn}^2(u) + \mathrm{dn}^2(u) = 1 \qquad \frac{d}{du}\,\mathrm{cn}(u) = -\,\mathrm{sn}(u)\,\mathrm{dn}(u)$$

$$\mathrm{sn}(0) = 0\ ;\ \mathrm{cn}(0) = \mathrm{dn}(0) = 1. \qquad \frac{d}{du}\,\mathrm{dn}(u) = -\,k^2\,\mathrm{sn}(u)\,\mathrm{cn}(u)$$

$$\mathrm{sn}(K) = 1\ ;\ \mathrm{cn}(K) = 0\ ;\ \mathrm{dn}(K) = \sqrt{1-k^2}$$

$$\mathrm{sn}(u+2K) = -\,\mathrm{sn}(u) \qquad (2K \text{ plays the role of } \pi)$$

$$\mathrm{cn}(u+2K) = -\,\mathrm{cn}(u)$$

$$\mathrm{dn}(u+2K) = \mathrm{dn}(u)$$

Our identity $(R+a)\cos\psi_1\cos\psi_2 + (R-a)\sin\psi_1\sin\psi_2 = r$ can be rewritten as: $\left(\text{recall } \dfrac{r}{R+a} = \cos(\mathrm{am}(c))\right)$

$$\mathrm{cn}(u_1)\,\mathrm{cn}(u_2) + \sqrt{1-k^2\,\mathrm{sn}^2(c)}\ \mathrm{sn}(u_1)\,\mathrm{sn}(u_2) = \mathrm{cn}(c)$$

$$\equiv \quad \mathrm{cn}(u_1)\,\mathrm{cn}(u_2) + \mathrm{dn}(c)\,\mathrm{sn}(u_1)\,\mathrm{sn}(u_2) = \mathrm{cn}(c)$$

The fact that $u_2 = u_1 + c$ solves this system follows from addition theorem (Abel):

$$\mathrm{sn}(z+\zeta) = \frac{\mathrm{sn}(z)\,\mathrm{cn}(\zeta)\,\mathrm{dn}(\zeta) + \mathrm{cn}(z)\,\mathrm{dn}(z)\,\mathrm{sn}(\zeta)}{1 - k^2\,\mathrm{sn}^2(z)\,\mathrm{sn}^2(\zeta)}$$

$$\mathrm{cn}(z+\zeta) = \frac{\mathrm{cn}(z)\,\mathrm{cn}(\zeta) - \mathrm{sn}(z)\,\mathrm{sn}(\zeta)\,\mathrm{dn}(z)\,\mathrm{dn}(\zeta)}{1 - k^2\,\mathrm{sn}^2(z)\,\mathrm{sn}^2(\zeta)}$$

References:

1. N.H. Abel. Recherches sur les fonctions elliptique (1827)
Journal für die reine und angwandte Mathematik (Crelle's journal).

2. A. Cayley. On the porism of the in-and-circumscribed triangle. (1857)
Quarterly Math. J.

3. V. Dragović and M. Radnović: Poncelet porisms & beyond (2011)
Frontiers in Math; Birkhäuser (Springer. Basel Switzerland)

4. P. Griffiths and J. Harris. On Cayley's explicit solution to Poncelet's porisms. (1978)
Enseign. Math.

5. C.G. Jacobi. Fundamenta nova theoriae functiorum ellipticarum. (1829)

6. H. Lebesgue. Les coniques (1942)

7. J.V. Poncelet. Traité des propriétés projectives des figures (1822)

8. H. Zoladek. The Poncelet theorems in interpretation of Rafal Kolodziej (2016)
Geometric methods in Physics.