WHAT IS The Collatz Conjecture?

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The Collatz Map

The Collatz Map $C : \mathbb{N} = \{1, 2, 3, \ldots\} \longrightarrow \mathbb{N}$ defined as

\[
C(n) = \begin{cases} 
3n + 1; & \text{if } n \text{ is odd} \\
\frac{n}{2}; & \text{if } n \text{ is even}
\end{cases}
\]  

(1)
The Collatz Conjecture - For every positive integer $n$, there exists an integer $k(n)$ such that $C^{k(n)}(n) = 1$.

- $4 \overset{C}{\rightarrow} 2 \overset{C}{\rightarrow} 1$
- $6 \overset{C}{\rightarrow} 3 \overset{C}{\rightarrow} 10 \overset{C}{\rightarrow} 5 \overset{C}{\rightarrow} 16 \overset{C}{\rightarrow} 8 \overset{C}{\rightarrow} 4 \overset{C}{\rightarrow} 2 \overset{C}{\rightarrow} 1$
Conjecture

The Collatz Conjecture - For every positive integer \( n \), there exists an integer \( k = k(n) \) such that \( C^k(n) = 1 \).
Sometimes the Collatz Conjecture is formulated in terms of the map $D : \mathbb{N} \rightarrow \mathbb{N}$ given by

$$D(n) = \begin{cases} \frac{3n+1}{2}; & \text{if } n \text{ is odd} \\ \frac{n}{2}; & \text{if } n \text{ is even} \end{cases} \quad (2)$$
Sometimes the Collatz Conjecture is formulated in terms of the map $D : \mathbb{N} \rightarrow \mathbb{N}$ given by

$$D(n) = \begin{cases} \frac{3n+1}{2}; & \text{if } n \text{ is odd} \\ n/2; & \text{if } n \text{ is even} \end{cases} \quad (2)$$

Note that $D(n) = \begin{cases} C(C(n)); & \text{if } n \text{ is odd} \\ C(n); & \text{if } n \text{ is even} \end{cases}$
Collatz Conjecture for $D(n)$

**The Collatz Conjecture:** For every positive integer $n$, there exists a $k = k(n)$ such that $D^k(n) = 1$.

For this interaction, both the cases will be referred as **The Collatz Conjecture**.
As a Graph.

Figure: Taken from [5]
Lothar Collatz and Friends.

- L. Collatz liked iterating number-theoretic functions and came up with quite a few problems like this conjecture.
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- Shared a lot of such problems through correspondences and word of mouth.
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Orbit of an Integer.

Given a positive integer $n$ we can create a sequence of values of its iterates

$$
\mathcal{O}(n) = \{n, C(n), C^2(n), C^3(n), C^4(n), \ldots\}
$$

and this is called orbit of $n$.

Here $C^2(n) = C(C(n))$ and $C^k(n) = C^{k-1}(n)$ for $k \geq 3$. 
Three Cases.

1. \( C^k(n) = 1 \) for some \( k \geq 1 \). ✓
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2. The orbit of iterates eventually gets stuck in a loop not containing 1. ✗
Three Cases.

1. $C^k(n) = 1$ for some $k \geq 1$. ✓

2. The orbit of iterates eventually gets stuck in a loop not containing 1. ❌

3. The orbit of iterates goes to infinity i.e. 
   $\lim_{k \to \infty} C^k(n) = +\infty$. ❌
How far have computations gone?

Verified Computationally for numbers $n > 10^{20}$.
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Figure: Taken From [14]
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Shalom Eliahou, in [4], proved that if any orbit gets stuck in a loop then the loop must have length at least GUESSES ??
Easy non-trivial Loop?

Shalom Eliahou, in [4], proved that if any orbit gets stuck in a loop then the loop must have length at least $17,087,915$. 
A Different Map

Consider the map $T(n) = \begin{cases} 
\frac{n}{2}; & \text{if } n \text{ is even} \\
3n - 1; & \text{if } n \text{ is odd}
\end{cases}$
A Different Map

Consider the map \( T(n) = \begin{cases} \frac{n}{2}; & \text{if } n \text{ is even} \\ 3n - 1; & \text{if } n \text{ is odd} \end{cases} \)

The orbit of 17 under \( T \) is a non-trivial loop

\[ 17 \rightarrow 50 \rightarrow 25 \rightarrow 74 \rightarrow 37 \rightarrow 110 \rightarrow 55 \rightarrow 164 \rightarrow 82 \rightarrow 41 \rightarrow 122 \rightarrow 61 \rightarrow 192 \rightarrow 91 \rightarrow 272 \rightarrow 136 \rightarrow 68 \rightarrow 34 \rightarrow 17. \]
Weak Collatz Conjecture.

The only periodic loop in the Collatz Conjecture is $4 \rightarrow 2 \rightarrow 1$. 
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Weak Collatz is equivalent to

There does not exist integers $0 = a_1 < a_2 < \ldots < a_{k+1}$ such that
$2^{a_{k+1}} - 3^k$ is a positive integer which divides

$$3^{k-1}2^{a_1} + 3^{k-2}2^{a_2} + \ldots + 2^{a_k}$$

See [10].
WHY THAT EQUIVALENCE?

- $48 \xrightarrow{c} 24 \xrightarrow{c} 12 \xrightarrow{c} 6 \xrightarrow{c} 3$

- $240 \xrightarrow{c} 120 \xrightarrow{c} 60 \xrightarrow{c} 30 \xrightarrow{c} 15$
WHY THAT EQUVALENCE?

- $\begin{align*}
48 & \rightarrow 24 & 24 & \rightarrow 12 & 12 & \rightarrow 6 & 6 & \rightarrow 3 \\
240 & \rightarrow 120 & 120 & \rightarrow 60 & 60 & \rightarrow 30 & 30 & \rightarrow 15 \\
2^k a & \rightarrow 2^{k-1} a & 2^{k-1} a & \rightarrow \ldots & \rightarrow 2a & 2a & \rightarrow a
\end{align*}$
Define an equivalence relation $\sim$ on $\mathbb{N}$ by $a \sim b$ if and only if $a = b \cdot 2^m$ for some $m \in \mathbb{Z}$.
To Jazz it Up.

Define an equivalence relation $\sim$ on $\mathbb{N}$ by $a \sim b$ if and only if $a = b \cdot 2^m$ for some $m \in \mathbb{Z}$.

Let $(\mathbb{N}/ \sim) = \{[n] : n \in \mathbb{N}\}$ be the set of equivalence classes, which can be identified with odd numbers.
Define an equivalence relation \( \sim \) on \( \mathbb{N} \) by \( a \sim b \) if and only if \( a = b \cdot 2^m \) for some \( m \in \mathbb{Z} \).

Let \((\mathbb{N}/\sim) = \{[n] : n \in \mathbb{N}\}\) be the set of equivalence classes, which can be identified with odd numbers.

Define \( E : \mathbb{N}/\sim \rightarrow \mathbb{N}/\sim \) as
\[
E([n]) = [3n + 2^a]
\]

Here \( 2^a \) is the highest power of 2 dividing \( n \).
The Map $E$ Continued.

- $E$ is well defined and only fixed point of the map $E$ is $[1]$. 

\[
E_k([n]) = \left[3^k n + 3^{k-1} + \ldots + 2^{a_k} \right]
\]
where $2^{a_i}$ is the highest power of 2 dividing 
\[
(3^i - 1^n + 3^i - 2^{a_1} + \ldots + 2^{a_i-1})
\]
The Map $E$ Continued.

- $E$ is well defined and only fixed point of the map $E$ is $[1]$.
- For any $n > 1$, we have

$$E^k([n]) = [3^k n + 3^{k-1} 2^{a_1} + \ldots + 2^{a_k}]$$

where $2^{a_i}$ is the highest power of 2 dividing

$$(3^{i-1} n + 3^{i-2} 2^{a_1} + \ldots + 2^{a_{i-1}}).$$
Proof of easy direction.

Assume that \( E^k([n]) = [n] \) for some \( k > 1 \) and \([n] \neq [1]\) with \( n \) odd.
Proof of easy direction.

Assume that \( E^k([n]) = [n] \) for some \( k > 1 \) and \([n] \neq [1]\) with \( n \) odd.

Recall that

\[
E^k([n]) = [3^kn + 3^{k-1}2^{a_1} + \ldots + 2^{a_k}]
\]

where \( 2^{a_i} \) is the highest power of 2 dividing

\[
n_i := (3^{i-1}n + 3^{i-2}2^{a_1} + \ldots + 2^{a_{i-1}}).
\]
Proof of easy direction.

Assume that \( E^k([n]) = [n] \) for some \( k > 1 \) and \( [n] \neq [1] \) with \( n \) odd.

Recall that

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E^k([n]) = [3^k n + 3^{k-1} 2^{a_1} + \ldots + 2^{a_k}]
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where \( 2^{a_i} \) is the highest power of 2 dividing

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n_i := (3^{i-1} n + 3^{i-2} 2^{a_1} + \ldots + 2^{a_{i-1}}).
\]

Note that \( n_1 = n, \ a_1 = 0 \) and

\[
n_{i+1} = 3n_i + 2^{a_i}.
\]
Since $2^{a_i} \mid n_i$ we have that $2^{a_i + 1} \mid n_{i+1}$. 

In other words, $(2^{a_k + 1} - 3^k)$ divides $3^k - 1$.
Since $2^{a_i} \mid n_i$ we have that $2^{a_i+1} \mid n_{i+1}$.

Therefore, $a_{i+1} \geq a_i + 1 > a_i$. 
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Therefore, $a_{i+1} \geq a_i + 1 > a_i$.

Since $E^k([n]) = [n]$ we have

$$2^{a_{k+1}}n = 3^k n + 3^{k-1}2^{a_1} + \ldots + 2^{a_k}$$

In other words, $(2^{a_{k+1}} - 3^k)$ divides $3^{k-1}2^{a_1} + \ldots + 2^{a_k}$.
Consequences of the Weak Collatz Conjecture.

- **Weak Collatz Conjecture** implies a very difficult result in transcendental number theory. [11]
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- Colloquially speaking, Alan Baker proved that gaps between powers of 2 and powers of 3 grow exponentially. [11]

- He received FIELDS MEDAL for associated works in 1970.
What Proportion of $n$ Satisfy the Conjecture.

R.E. Crandall, in 1978, first showed that the proportion of natural numbers in \{1, 2, \ldots, N\} that satisfy the conjecture is about $C \cdot N^\gamma$, for some constant $\gamma > 0$. [3]

D. Applegate and J.C. Lagarias, in 1995, showed that $\gamma > 0.81$. [1]

Krasikov and Lagarias, in 2003, showed that $\gamma > 0.84$, which is state of the art. [6]
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Ideally.

We would like to prove that for every $n$, there is a $k$ such that $g^k(n) = 1$ for some $k$. 
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A preliminary step is to show that:
There is a $M > 0$ such that for every $n$ there is a $k$ such that $g^k(n) \leq M$. 

Even above seems out of reach currently. So, what can we prove?
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- Even above seems out of reach currently.

- So, what can we prove? ⇒ Some observations.
Heuristic Reasoning.

The map $D : \mathbb{N} \rightarrow \mathbb{N}$ given by

$$D(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

increases an odd number by $\approx \frac{3}{2}$ and decreases an even number by $\frac{1}{2}$. [10]
If we could choose \( n \) with equal probability of being an odd or even then

\[
D(n) \approx \frac{3}{2} n \text{ with probability } \frac{1}{2}
\]

\[
D(n) = \frac{1}{2} n \text{ with probability } \frac{1}{2}
\]
If $D(n)$ were even and odd with equal probability.

\[
D(n) \text{ ODD} \implies D^2(n) \approx \frac{3}{2} D(n)
\]

\[
D(n) \text{ EVEN} \implies D^2(n) = \frac{1}{2} D(n)
\]

with equal probability $\frac{1}{2}$. 
Similarly...

**ODD:** $D^k(n) \approx \frac{3}{2} D^{k-1}(n) \iff \log(D^k(n)) \approx \log \frac{3}{2} + \log(D^{k-1}(n))$

**EVEN:** $D^k(n) = \frac{1}{2} D^{k-1}(n) \iff \log(D^k(n)) \approx \log \frac{1}{2} + \log(D^{k-1}(n))$
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$\{\log(n), \log(D(n)), \log(D^2(n)), \ldots\} \approx$ random walk with 2 possible steps of size $\log(\frac{3}{2})$ and $\log(\frac{1}{2})$ respectively, each can be taken with probability $\frac{1}{2}$. 
Final Episode of this Ideal Heuristic.

The expectation of such random walk is

\[ \frac{1}{2} \log\left(\frac{3}{2}\right) + \frac{1}{2} \log\left(\frac{1}{2}\right) = \frac{1}{2} \log\left(\frac{3}{4}\right) \approx -0.1438 < 0 \]
The expectation of such random walk is

\[
\frac{1}{2} \log\left(\frac{3}{2}\right) + \frac{1}{2} \log\left(\frac{1}{2}\right) = \frac{1}{2} \log\left(\frac{3}{4}\right) \approx -0.1438 < 0
\]

So we expect this random walk to decrease the value in the long run i.e. for some \( k, D^k(n) < n \). [10] [ THINK GAMBLER’S RUIN ]
Progress on Proving what Heuristic Suggests.

- Given $A \subset \mathbb{N}$, we define density of $A$ as

$$d(A) = \lim_{N \to \infty} \frac{|A \cap \{1, 2, \ldots, N\}|}{N}$$

if the limit exists.
Progress on Proving what Heuristic Suggests.

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$d(\mathbb{N}) = 1$

$d(\mathbb{N} \setminus \{ \text{finite set } \}) = 1$

$d(\text{Primes}) = 0$
Progress on Proving what Heuristic Suggests.

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if the limit exists.

- $d(\mathbb{N}) = 1$
- $d(\mathbb{N} \setminus \{\text{finite set}\}) = 1$
- $d(\text{Primes}) = 0$

We say that a property $P$ holds for almost every natural number exactly when the set of $n \in \mathbb{N}$ that do not satisfy $P$ has density zero.
Given any $n \in \mathbb{N}$, define

$$Min(n) := \min_{k \geq 1} D^k(n)$$
Given any \( n \in \mathbb{N} \), define

\[
\text{Min}(n) := \min_{k \geq 1} D^k(n)
\]

Riho Terras (1976) showed that for almost every \( n \) we have

\[
\text{Min}(n) < n.
\]

[13]
Improving on Terras.

\[
\text{Min}(n) < \begin{cases} 
    n^{0.869} & ; [2] \\
    n^{0.7924} & ; [7]
\end{cases}
\]
For almost every $n$ and for any function $f$ such that \( \lim_{n \to \infty} f(n) = \infty \),

\[
\text{Min} (n) < f(n)
\]
Massive Improvement by Terrence Tao.

- For almost every \( n \) and for any function \( f \) such that
  \[
  \lim_{n \to \infty} f(n) = \infty,
  \]
  \[\text{Min}(n) < f(n)\]

  \[[12]\]

- So, \( \text{Min}(n) < n^{0.000001} \)

\[\text{Min}(n) < \log(\log(\ldots(\log(n))\ldots))\].
Summary.

Collatz Conjecture has two distinct parts: Non-trivial loops and Boundedness of Orbit.
Summary.

- Collatz Conjecture has two distinct parts: **Non-trivial loops**
  Boundedness of Orbit.

- Showing **NO NON-TRIVIAL LOOPS EXISTS**, appears out of
  reach for now.

- Showing that orbit is bounded is also out of reach, as per
  current mathematical technology.
Summary.

- Collatz Conjecture has two distinct parts: **Non-trivial loops** and **Boundedness of Orbit**.

- Showing **NO NON-TRIVIAL LOOPS EXISTS**, appears out of reach for now.

- Showing that orbit is bounded is also out of reach, as per current mathematical technology.

- Most works aimed to show that Min(n) is bounded by some function of n.

- T. Tao made the most significant improvement on this problem, so far.
Some Tweaks of Collatz Conjecture.

\[ g(n) = \begin{cases} 
\frac{2n}{3} & \text{if } n \equiv 0 \pmod{3} \\
\frac{4n-1}{3} & \text{if } n \equiv 1 \pmod{3} \\
\frac{4n+1}{3} & \text{if } n \equiv 2 \pmod{3}
\end{cases} \]
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\end{cases}$$

This is a permutation of \( \mathbb{N} \) given by \((1) (2 \ 3) (4 \ 5 \ 7 \ 9 \ 6) \ldots \).
Some Tweaks of Collatz Conjecture.

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- This is a permutation of \( \mathbb{N} \) given by \((1) (2 3) (4 5 7 9 6) \ldots \).
- Is the set \( \{g^k(8)\}_{k \geq 1} \) finite?
Some Tweaks of Collatz Conjecture.

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\end{cases} \]

- This is a permutation of \( \mathbb{N} \) given by \( (1) (2 3) (4 5 7 9 6) \ldots \).
- Is the set \( \{g^k(8)\}_{k \geq 1} \) finite?
- In other words, is the cycle containing 8 finite?
The $5n + 1$ Version.

\[
h(n) = \begin{cases} 
\frac{5n+1}{2} & \text{if } n \equiv 1 \pmod{2} \\
\frac{n}{2} & \text{if } n \equiv 0 \pmod{2}
\end{cases}
\]
The $5n + 1$ Version.

\[ h(n) = \begin{cases} \frac{5n+1}{2} & \text{if } n \equiv 1 \pmod{2} \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \end{cases} \]

Widely believed that for almost every $n$, 
\[ \lim_{k \to \infty} h^k(n) = +\infty. \]
The $pn + 1$ Conjecture.

Let $p > 3$ be an odd number and define

$$h_p(n) = \begin{cases} \frac{pn+1}{2} ; & \text{if } n \text{ is odd} \\ \frac{n}{2} ; & \text{if } n \text{ is even} \end{cases}$$

Conjecture - For almost every $n$, $\lim_{k \in \mathbb{N}} h_p^k(n) = \infty$. 
Thank you for listening.


Korec, I. A density estimate for the 3x + 1 problem. Math. Slovaca 44 (1). 85–89.


https://xkcd.com/710/.