WHAT IS **The Collatz Conjecture** ?

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Contents.

The Collatz Map

The Collatz Map $C : \mathbb{N} = \{1, 2, 3, \ldots\} \longrightarrow \mathbb{N}$ defined as

$$C(n) = \begin{cases} 3n + 1; \text{ if n is odd} \\ \frac{n}{2}; \text{ if n is even} \end{cases} \tag{1}$$

Conjecture

- $4 \xrightarrow{C} 2 \xrightarrow{C} 1$
- $6 \xrightarrow{C} 3 \xrightarrow{C} 10 \xrightarrow{C} 5 \xrightarrow{C} 16 \xrightarrow{C} 8 \xrightarrow{C} 4 \xrightarrow{C} 2 \xrightarrow{C} 1$

Conjecture

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- $6 \xrightarrow{C} 3 \xrightarrow{C} 10 \xrightarrow{C} 5 \xrightarrow{C} 16 \xrightarrow{C} 8 \xrightarrow{C} 4 \xrightarrow{C} 2 \xrightarrow{C} 1$
- The Collatz Conjecture - For every positive integer n, there exists an integer $k = k(n)$ such that $C^k(n) = 1$.

Slight Modification.

Sometimes the Collatz Conjecture is formulated in terms of the map $D : \mathbb{N} \longrightarrow \mathbb{N}$ given by

$$D(n) = \begin{cases} \frac{3n+1}{2}; \text{ if n is odd} \\ \frac{n}{2}; \text{ if n is even} \end{cases} \tag{2}$$

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Note that $D(n) = \begin{cases} C(C(n)); \text{if n is odd} \\ C(n); \text{ if n is even} \end{cases}$

Collatz Conjecture for $D(n)$

The Collatz Conjecture: For every positive integer n, there exists a $k = k(n)$ such that $D^k(n) = 1$.

For this interaction, both the cases will be referred as **The Collatz Conjecture**.

As a Graph.

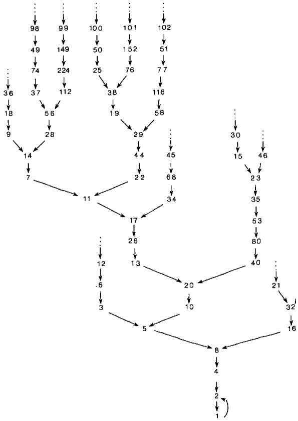

Fig. 1. The Collatz graph.

Figure: Taken from [5]

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- Shared a lot of such problems through correspondences and word of mouth.
- S. Kakutani, H. S. M. Coxeter, S. Ulam, H. Hasse, J. H. Conway, R. Guy etc. helped spread this problem around.

Orbit of an Integer.

Given a positive integer n we can create a sequence of values of its iterates

$$\mathcal{O}(n) = \{n, C(n), C^2(n), C^3(n), C^4(n), \ldots\}$$

and this is called *orbit* of n.

Here $C^2(n) = C(C(n))$ and $C^k(n) = C^{k-1}(n)$ for $k \geq 3$.

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How far have computations gone ?

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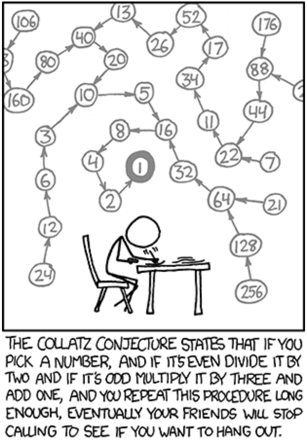

Figure: Taken From [14]

Easy non-trivial Loop ?

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$$17,087,915$$

A Different Map

Consider the map $T(n) = \begin{cases} \frac{n}{2}; \text{ if n is even} \\ 3n - 1; \text{ if n is odd} \end{cases}$

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The orbit of 17 under T is a non-trivial loop

$17 \to 50 \to 25 \to\to 74 \to 37 \to 110 \to 55 \to 164 \to 82 \to 41 \to 122 \to 61 \to 192 \to 91 \to 272 \to 136 \to 68 \to 34 \to 17.$

Weak Collatz Conjecture.

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Weak Collatz is equivalent to

There does not exist integers $0 = a_1 < a_2 < \ldots < a_{k+1}$ such that $2^{a_{k+1}} - 3^k$ is a positive integer which divides

$$3^{k-1}2^{a_1} + 3^{k-2}2^{a_2} + \ldots + 2^{a_k}$$

See [10].

WHY THAT EQUIVALENCE ?

- $48 \xrightarrow{C} 24 \xrightarrow{C} 12 \xrightarrow{C} 6 \xrightarrow{C} 3$
- $240 \xrightarrow{C} 120 \xrightarrow{C} 60 \xrightarrow{C} 30 \xrightarrow{C} 15$

WHY THAT EQUIVALENCE ?

- $48 \xrightarrow{C} 24 \xrightarrow{C} 12 \xrightarrow{C} 6 \xrightarrow{C} 3$
- $240 \xrightarrow{C} 120 \xrightarrow{C} 60 \xrightarrow{C} 30 \xrightarrow{C} 15$
- $2^k a \xrightarrow{C} 2^{k-1} a \xrightarrow{C} \cdots \xrightarrow{C} 2a \xrightarrow{C} a$

To Jazz it Up.

- Define an equivalence relation $\sim$ on $\mathbb{N}$ by $a \sim b$ if and only if $a = b \cdot 2^m$ for some $m \in \mathbb{Z}$.

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- Let $(\mathbb{N}/\sim) = \{[n] : n \in \mathbb{N}\}$ be the set of equivalence classes , which can be identified with odd numbers.
- Define $E : \mathbb{N}/\sim \longrightarrow \mathbb{N}/\sim$ as

$$E([n]) = [3n + 2^a]$$

. Here 2^a is the highest power of 2 dividing n.

The Map E Continued.

- E is well defined and only fixed point of the map E is [1].

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- E is well defined and only fixed point of the map E is [1].
- For any $n > 1$, we have

$$E^k([n]) = [3^k n + 3^{k-1} 2^{a_1} + \ldots + 2^{a_k}]$$

where 2^{a_i} is the highest power of 2 dividing

$$(3^{i-1} n + 3^{i-2} 2^{a_1} + \ldots + 2^{a_{i-1}}).$$

Proof of easy direction.

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$$n_i := (3^{i-1}n + 3^{i-2}2^{a_1} + \ldots + 2^{a_{i-1}}).$$

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$$n_i := (3^{i-1}n + 3^{i-2}2^{a_1} + \ldots + 2^{a_{i-1}}).$$

Note that $n_1 = n$, $a_1 = 0$ and

$$n_{i+1} = 3n_i + 2^{a_i}.$$

- Since $2^{a_i} \mid n_i$ we have that $2^{a_i+1} \mid n_{i+1}$.

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- Therefore, $a_{i+1} \geq a_i + 1 > a_i$.
- Since $E^k([n]) = [n]$ we have

$$2^{a_{k+1}} n = 3^k n + 3^{k-1} 2^{a_1} + \ldots + 2^{a_k}$$

- In other words, $(2^{a_{k+1}} - 3^k)$ divides $3^{k-1} 2^{a_1} + \ldots + 2^{a_k}$.

Consequences of the Weak Collatz Conjecture.

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- Weak Collatz Conjecture implies a very difficult result in transcendental number theory. [11]
- Colloquially speaking, Alan Baker proved that gaps between powers of 2 and powers of 3 grow *exponentially*. [11]
- He received FIELDS MEDAL for associated works in 1970.

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- D. Applegate and J.C. Lagarias, in 1995, showed that $\gamma > 0.81$. [1]
- Krasikov and Lagarias, in 2003, showed that $\gamma > 0.84$, which is state of the art. [6]

Recall this Slide ?

1. $C^k(n) = 1$ for some $k \geq 1$. ✓
2. The orbit of iterates eventually gets stuck in a loop not containing 1. ✗
3. The orbit of iterates goes to infinity i.e. $\lim_{k\to\infty} C^k(n) = +\infty$. ✗

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- Even above seems out of reach currently.
- So, what can we prove ? $\Rightarrow$ Some observations.

Heuristic Reasoning.

The map $D : \mathbb{N} \longrightarrow \mathbb{N}$ given by

$$D(n) = \begin{cases} \frac{3n+1}{2}; \text{ if n is odd} \\ \frac{n}{2}; \text{ if n is even} \end{cases} \tag{3}$$

increases an odd number by $\approx \frac{3}{2}$ and decreases an even number by $\frac{1}{2}$. [10]

Naive Hope.

- If we could choose n with equal probability of being an odd or even then

$$D(n) \approx \frac{3}{2}n \text{ with probability } \frac{1}{2}$$

$$D(n) = \frac{1}{2}n \text{ with probability} \frac{1}{2}$$

If $D(n)$ were even and odd with equal probability.

$$D(n) \text{ ODD} \implies D^2(n) \approx \frac{3}{2} D(n)$$

$$D(n) \text{ EVEN} \implies D^2(n) = \frac{1}{2} D(n)$$

with equal probability $\frac{1}{2}$.

Similarly...

ODD: $D^k(n) \approx \frac{3}{2} D^{k-1}(n) \iff \log(D^k(n)) \approx \log \frac{3}{2} + \log(D^{k-1}(n))$

EVEN: $D^k(n) = \frac{1}{2} D^{k-1}(n) \iff \log(D^k(n)) \approx \log \frac{1}{2} + \log(D^{k-1}(n))$

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$\{\log(n), \log(D(n)), \log(D^2(n)), \ldots\} \approx$ random walk with 2 possible steps of size $\log(\frac{3}{2})$ and $\log(\frac{1}{2})$ respectively, each can be taken with probability $\frac{1}{2}$.

Final Episode of this Ideal Heuristic.

The expectation of such random walk is

$$\frac{1}{2}\log(\frac{3}{2}) + \frac{1}{2}\log(\frac{1}{2}) = \frac{1}{2}\log(\frac{3}{4}) \approx -0.1438 < 0$$

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The expectation of such random walk is

$$\frac{1}{2}\log(\frac{3}{2}) + \frac{1}{2}\log(\frac{1}{2}) = \frac{1}{2}\log(\frac{3}{4}) \approx -0.1438 < 0$$

So we expect this random walk to decrease the value in the long run i.e. for some k, $D^k(n) < n$. [10] $\Big[$ THINK GAMBLER'S RUIN $\Big]$

Progress on Proving what Heuristic Suggests.

- Given $A \subset \mathbb{N}$, we define density of A as

$$d(A) = \lim_{N \to \infty} \frac{|\{A \cap \{1, 2, , \dots, N\}|}{N}$$

if the limit exists.

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- We say that a property P holds for almost every natural number exactly when the set of $n \in \mathbb{N}$ that do not satisfy P has density zero.

- Given any $n \in \mathbb{N}$, define

$$Min(n) := \min_{k \geq 1} D^k(n)$$

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- Riho Terras (1976) showed that for almost every n we have

$$\mathrm{Min(n)} \; < n.$$

[13]

Improving on Terras.

$$\text{Min(n)} < \begin{cases} n^{0.869} & ; [2] \\ n^{0.7924} & ; [7] \end{cases}$$

Massive Improvement by Terrence Tao.

- For almost every n and for any function f such that $\lim_{n\to\infty} f(n) = \infty$,

$$\text{Min}\ (n) < f(n)$$

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- For almost every n and for any function f such that $\lim_{n\to\infty} f(n) = \infty$,

  $$\text{Min }(n) < f(n)$$

  [12]

- So, $\text{Min}(n) < n^{0.000001}$

  $\text{Min}(n) < \log(\log(\ldots(\log(n))\ldots))$.

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Summary.

- Collatz Conjecture has two distinct parts : Non-trivial loops Boundedness of Orbit.
- Showing NO NON-TRIVIAL LOOPS EXISTS, appears out of reach for now.
- Showing that orbit is bounded is also out of reach, as per current mathematical technology.
- Most works aimed to show that $\mathrm{Min}(n)$ is bounded by some function of n.
- T. Tao made the most significant improvement on this problem, so far.

Some Tweaks of Collatz Conjecture.

$$g(n) = \begin{cases} \frac{2n}{3}\ ;\ \text{if } n \equiv 0 \pmod 3 \\ \frac{4n-1}{3}\ ;\ \text{if } n \equiv 1 \pmod 3 \\ \frac{4n+1}{3}\ ;\ \text{if } n \equiv 2 \pmod 3 \end{cases}$$

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- This is a permutation of $\mathbb{N}$ given by (1) (2 3) (4 5 7 9 6) . . . .

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- This is a permutation of $\mathbb{N}$ given by (1) (2 3) (4 5 7 9 6) . . . .
- Is the set $\{g^k(8)\}_{k \geq 1}$ finite ?
- In other words, is the cycle containing 8 finite?

The $5n + 1$ Version.

$$h(n) = \begin{cases} \frac{5n+1}{2} \ ; \ \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} \ ; \ \text{if } n \equiv 0 \pmod 2 \end{cases}$$

The $5n+1$ Version.

$$h(n) = \begin{cases} \frac{5n+1}{2}\ ;\ \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2}\ ;\ \text{if } n \equiv 0 \pmod 2 \end{cases}$$

- Widely believed that for almost every n, $\lim_{k\to\infty} h^k(n) = +\infty$.

The $pn+1$ Conjecture.

Let $p>3$ be an odd number and define

$$h_p(n)=\begin{cases}\frac{pn+1}{2}\ ;\ \text{if n is odd}\\ \frac{n}{2}\ ;\ \text{if n is even}\end{cases}$$

Conjecture - For almost every n, $\lim_{k\in\mathbb{N}} h_p^k(n)=\infty$.

Thank you for listening.

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