What are.... some nice conjectures in Graph Theory?

What is Seminar 2021

Sohail Farhangi

June 24, 2021

Overview

Table of Contents

Cayley Graphs

Definition

Given a group G and a set $S \subseteq G$, the Cayley Graph $\mathcal{T} = \mathcal{T}(G, S) = (V, E)$ is defined by $V = G$, and $(g_1, g_2) \in E$ iff $g_2 g_1^{-1} \in S$.

Cayley Graphs

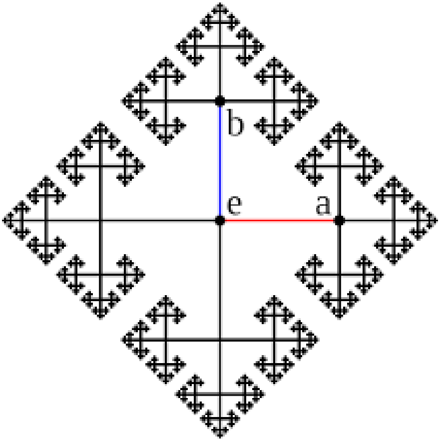
Definition

Given a group G and a set $S \subseteq G$, the Cayley Graph $\mathcal{T} = \mathcal{T}(G, S) = (V, E)$ is defined by $V = G$, and $(g_1, g_2) \in E$ iff $g_2 g_1^{-1} \in S$.

In this talk we will usually assume that G is finite, $S = S^{-1}$, and S generates G. This will result in a finite connected undirected graph.

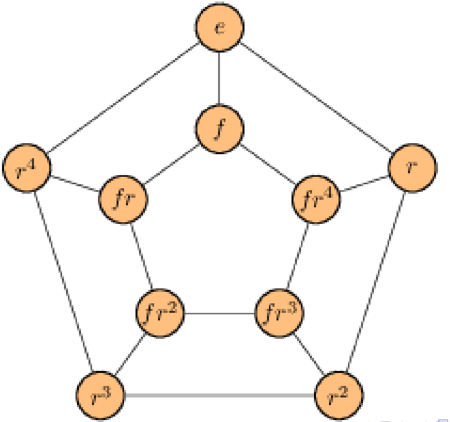
A Cayley Graph of F_2

Letting F_2 denote the free group generated by a and b, we see that $\mathcal{T}(F_2, \{a, b, a^{-1}, b^{-1}\})$ is given by



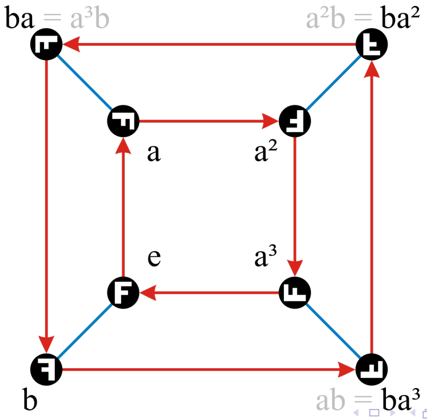
A Cayley Graph of D_5

Letting $D_5 = \langle f, r \rangle$ denote the group of symmetries of a regular pentagon with f for reflection and r for rotation, we see that $\mathcal{T}(D_5, \{f, r, r^{-1}\})$ is given by



A Cayley Digraph of D_4

Letting $D_4 = \langle a, b \rangle$ denote the group of symmetries of a square with b for reflection and a for rotation, we see that $\mathcal{T}(D_5, \{a, b\})$ is given by



The Lovász Conjecture (version 1)

Conjecture

The Cayley graph of a finite group has a Hamiltonian cycle.

Known Results (also cf. [4])

cf. [11] Every finite group G has a generating set S of size at most $\log_2(|G|)$ for which the Cayley graph $\mathcal{T}(G, S)$ is Hamiltonian.

Known Results (also cf. [4])

cf. [11] Every finite group G has a generating set S of size at most $\log_2(|G|)$ for which the Cayley graph $\mathcal{T}(G, S)$ is Hamiltonian.

cf. [14] The Lovász conjecture holds for Cayley digraphs of p-groups.

Known Results (also cf. [4])

cf. [11] Every finite group G has a generating set S of size at most $\log_2(|G|)$ for which the Cayley graph $\mathcal{T}(G, S)$ is Hamiltonian.

cf. [14] The Lovász conjecture holds for Cayley digraphs of p-groups.

- The Lovász conjecture is still open for dihedral groups.

Known Results (also cf. [4])

cf. [11] Every finite group G has a generating set S of size at most $\log_2(|G|)$ for which the Cayley graph $\mathcal{T}(G, S)$ is Hamiltonian.

cf. [14] The Lovász conjecture holds for Cayley digraphs of p-groups.

- The Lovász conjecture is still open for dihedral groups.
- The Lovász conjecture is false for Cayley digraphs.

Known Results (also cf. [4])

cf. [11] Every finite group G has a generating set S of size at most $\log_2(|G|)$ for which the Cayley graph $\mathcal{T}(G, S)$ is Hamiltonian.

cf. [14] The Lovász conjecture holds for Cayley digraphs of p-groups.

- The Lovász conjecture is still open for dihedral groups.
- The Lovász conjecture is false for Cayley digraphs.

cf. [7] Every Cayley digraph of an abelian group has a Hamiltonian path.

Known Results (also cf. [4])

cf. [11] Every finite group G has a generating set S of size at most $\log_2(|G|)$ for which the Cayley graph $\mathcal{T}(G, S)$ is Hamiltonian.

cf. [14] The Lovász conjecture holds for Cayley digraphs of p-groups.

- The Lovász conjecture is still open for dihedral groups.
- The Lovász conjecture is false for Cayley digraphs.

cf. [7] Every Cayley digraph of an abelian group has a Hamiltonian path.

cf. [7] Every cyclic group whose order is not a prime power has a Cayley digraph that does not have a Hamiltonian cycle.

Known Results (also cf. [4])

cf. [5] Let p and q be primes. A Cayley graph on a group of order $pq, 4q (q > 3), p^2q (2 < p < q), 2p^2, 2pq, 8p$, or $4p^2$ is Hamiltonian.

Known Results (also cf. [4])

cf. [5] Let p and q be primes. A Cayley graph on a group of order $pq, 4q(q > 3), p^2q(2 < p < q), 2p^2, 2pq, 8p$, or $4p^2$ is Hamiltonian.

cf. [6] If $1 \leq k \leq 31$ and $k \neq 24$, then for any prime p, any group G of order kp satisfies the Lovász conjecture.

The Lovász Conjecture (version 2)

Definition

A graph $G = (V, E)$ is **vertex-transitive** if for any $v_1, v_2 \in V$, there exists a graph automorphism $\tau : G \to G$ for which $\tau(v_1) = v_2$.

The Lovász Conjecture (version 2)

Definition

A graph $G = (V, E)$ is **vertex-transitive** if for any $v_1, v_2 \in V$, there exists a graph automorphism $\tau : G \to G$ for which $\tau(v_1) = v_2$.

Example

Any (general) Cayley graph $(T)(G, S)$ is vertex-transitive.

The Lovász Conjecture (version 2)

Definition

A graph $G = (V, E)$ is **vertex-transitive** if for any $v_1, v_2 \in V$, there exists a graph automorphism $\tau : G \to G$ for which $\tau(v_1) = v_2$.

Example

Any (general) Cayley graph $(\mathcal{T})(G, S)$ is vertex-transitive.

To see this, we just note that for any $g_1, g_2 \in G$, the map given by $\tau(g) = g_2 g_1^{-1} g$ is an automorphism of $\mathcal{T}$ satisfying $\tau(g_1) = g_2$.

The Lovász Conjecture (version 2)

Definition

A graph $G = (V, E)$ is **vertex-transitive** if for any $v_1, v_2 \in V$, there exists a graph automorphism $\tau : G \to G$ for which $\tau(v_1) = v_2$.

Example

Any (general) Cayley graph $(T)(G, S)$ is vertex-transitive.

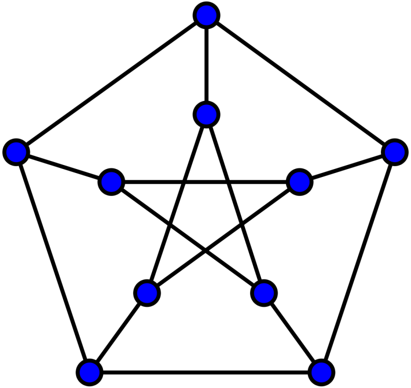
To see this, we just note that for any $g_1, g_2 \in G$, the map given by $\tau(g) = g_2 g_1^{-1} g$ is an automorphism of $\mathcal{T}$ satisfying $\tau(g_1) = g_2$.

Conjecture

Every finite vertex-transitive graph contains a Hamiltonian path.

The Petersen Graph

We see that the Petersen graph is vertex transitive and contains a Hamiltonian path, but does not contain a Hamiltonian cycle.



The Lovász Conjecture (version 3)

Conjecture

Every finite connected vertex-transitive graph has a Hamiltonian cycle except for the 5 known counterexamples.

The 5 Counterexamples

(1) K_2, the complete graph on 2 vertices.

The 5 Counterexamples

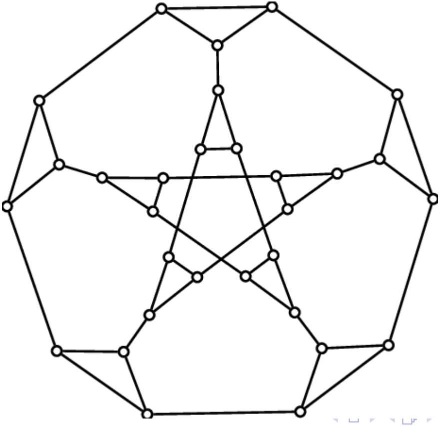
(1) K_2, the complete graph on 2 vertices.
(2) The Petersen graph P.

The 5 Counterexamples

(1) K_2, the complete graph on 2 vertices.

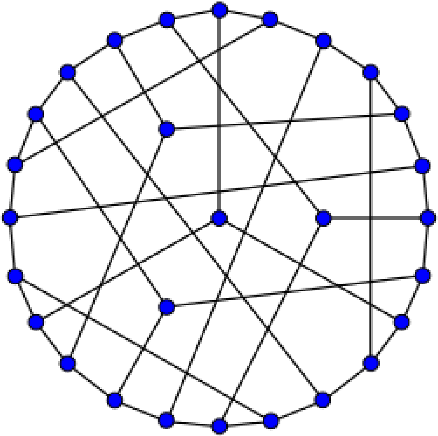
(2) The Petersen graph P.

(3) The Petersen graph with all vertices replaced by triangles.



The 5 Counterexamples

(4) The Coxeter graph.



The 5 Counterexamples

(4) The Coxeter graph with all vertices replaced by triangles.

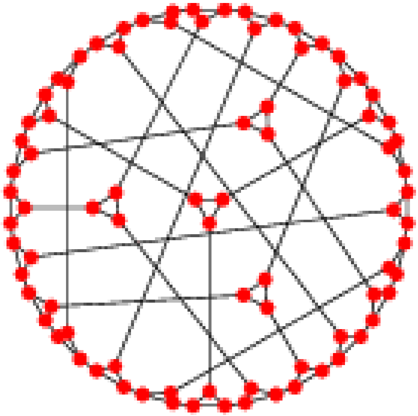


Table of Contents

edge-weighting vertex colourings

Definition

An **edge-weighting vertex colouring** of a graph $G = (V, E)$ is an edge-weighting assignment $f : E \to \mathbb{R}$ such that the accumulated weights at the vertices yields a proper vertex-colouring,

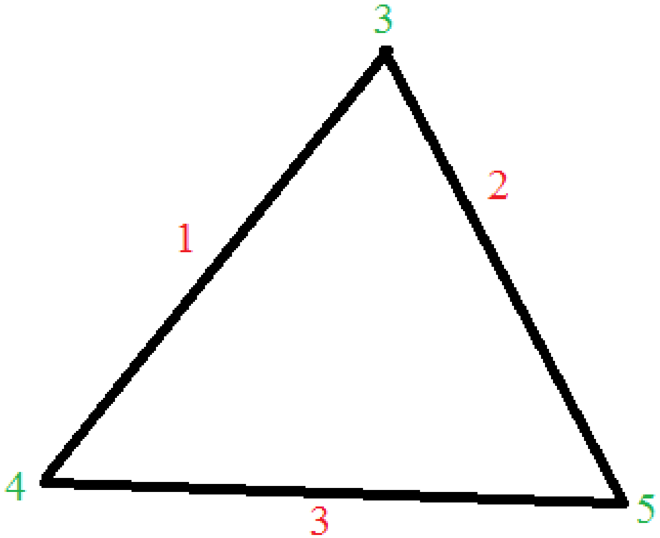
edge-weighting vertex colourings

Definition

An **edge-weighting vertex colouring** of a graph $G = (V, E)$ is an edge-weighting assignment $f : E \to \mathbb{R}$ such that the accumulated weights at the vertices yields a proper vertex-colouring,where the accumulated weight at a vertex v is given by

$$a(v) = \sum_{(v,v')\in E} f(v, v'). \tag{1}$$

An Example

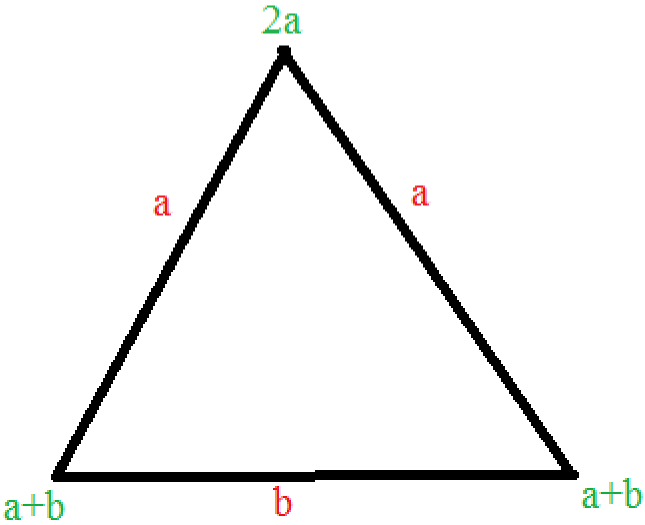


The 1-2-3 Conjecture

Conjecture (M. Karoński, T. Łuczak, A. Thomason (2004, cf. [3]))

For any finite connected graph $G = (V, E)$ that is not K_2, there exists an edge-weighting assignment $f : E \rightarrow \{1, 2, 3\}$ which results in an edge-weighting vertex colouring.

The 1-2 Conjecture is False



Known Results

- The 1-2-3 Conjecture was proven for 3-colorable graphs by M. Karoński, T. Łuczak, and A. Thomason in 2004. (cf. [10])

Known Results

- The 1-2-3 Conjecture was proven for 3-colorable graphs by M. Karoński, T. Łuczak, and A. Thomason in 2004. (cf. [10])
- The 1-2-$\cdots$-30 Conjecture was proven by L. Addario-Berry, K. Dalal, C. McDiarmid, B.A. Reed, and A. Thomason in 2007. (12 pages, cf. [1])

Known Results

- The 1-2-3 Conjecture was proven for 3-colorable graphs by M. Karoński, T. Łuczak, and A. Thomason in 2004. (cf. [10])
- The 1-2-$\cdots$-30 Conjecture was proven by L. Addario-Berry, K. Dalal, C. McDiarmid, B.A. Reed, and A. Thomason in 2007. (12 pages, cf. [1])
- The 1-2-$\cdots$-16 Conjecture was proven by L. Addario-Berry, K. Dalal, and B.A. Reed in 2008. (7 pages, cf. [2])

Known Results

- The 1-2-3 Conjecture was proven for 3-colorable graphs by M. Karoński, T. Łuczak, and A. Thomason in 2004. (cf. [10])
- The 1-2-$\cdots$-30 Conjecture was proven by L. Addario-Berry, K. Dalal, C. McDiarmid, B.A. Reed, and A. Thomason in 2007. (12 pages, cf. [1])
- The 1-2-$\cdots$-16 Conjecture was proven by L. Addario-Berry, K. Dalal, and B.A. Reed in 2008. (7 pages, cf. [2])
- The 1-2-$\cdots$-13 Conjecture was proven by by T. Wang and Q. Yu in 2008. (7 pages, cf. [13])

Known Results

- The 1-2-3 Conjecture was proven for 3-colorable graphs by M. Karoński, T. Łuczak, and A. Thomason in 2004. (cf. [10])
- The 1-2-$\cdots$-30 Conjecture was proven by L. Addario-Berry, K. Dalal, C. McDiarmid, B.A. Reed, and A. Thomason in 2007. (12 pages, cf. [1])
- The 1-2-$\cdots$-16 Conjecture was proven by L. Addario-Berry, K. Dalal, and B.A. Reed in 2008. (7 pages, cf. [2])
- The 1-2-$\cdots$-13 Conjecture was proven by by T. Wang and Q. Yu in 2008. (7 pages, cf. [13])
- The 1-2-3-4-5-6 Conjecture was proven by M. Kalkowski, M. Karoński, and F. Pfender in 2009. (5 pages, cf. [8])

Known Results

- The 1-2-3-4-5 Conjecture was proven by M. Kalkowski, M. Karoński, and F. Pfender in 2010. (3 pages, cf. [9])

Known Results

- The 1-2-3-4-5 Conjecture was proven by M. Kalkowski, M. Karoński, and F. Pfender in 2010. (3 pages, cf. [9])
- The 1-2-3-4 Conjecture for regular graphs and the 1-2-3 Conjecture for d-regular graphs with $d \geq 10^8$ was proven by J. Przybyło in 2021. (cf. [12])

The Oriented 1-2-3 "Conjecture"

Definition

An **edge-weighting vertex colouring** of an oriented graph $G = (V, E)$ is an edge-weighting assignment $f : E \to \mathbb{R}$ such that the accumulated weights at the vertices yields a proper vertex-colouring,

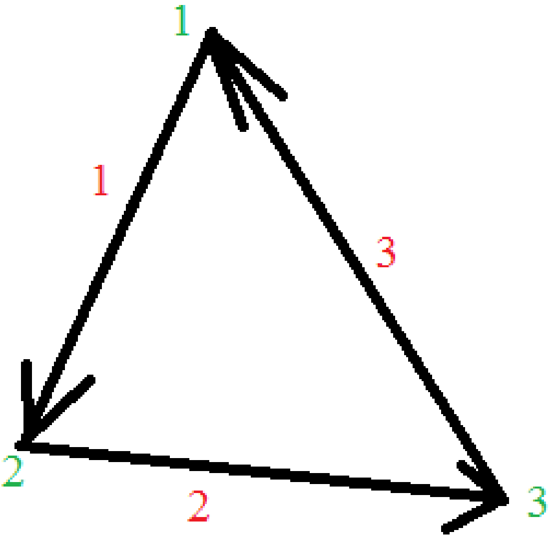
The Oriented 1-2-3 “Conjecture”

Definition

An **edge-weighting vertex colouring** of an oriented graph $G = (V, E)$ is an edge-weighting assignment $f : E \rightarrow \mathbb{R}$ such that the accumulated weights at the vertices yields a proper vertex-colouring,where the accumulated weight at a vertex v is given by

$$a(v) = \sum_{(v,v')\in E} f(v, v'). \tag{2}$$

An Example



The Oriented 1-2-3 Theorem

Theorem (O. Baudon, J. Bensmail, E. Sopena (2015, cf. [3]))

Every oriented graph $G = (V, E)$ admits an edge-weighting vertex colouring with an edge-weighting assignment $f : E \to \{1, 2, 3\}$.

Bibliography I

[1] L. Addario-Berry, K. Dalal, C. McDiarmid, B. A. Reed, and A. Thomason.
Vertex-colouring edge-weightings.
Combinatorica, 27(1):1–12, 2007.

[2] L. Addario-Berry, K. Dalal, and B. A. Reed.
Degree constrained subgraphs.
Discrete Appl. Math., 156(7):1168–1174, 2008.

[3] O. Baudon, J. Bensmail, and E. Sopena.
An oriented version of the 1-2-3 conjecture.
Discuss. Math. Graph Theory, 35(1):141–156, 2015.

Bibliography II

[4] S. J. Curran and J. A. Gallian.
Hamiltonian cycles and paths in Cayley graphs and digraphs—a survey.
Discrete Math., 156(1-3):1–18, 1996.

[5] E. Friedman, D. Jungreis, and D. Witte.
Cayley graphs on groups of low order are hamiltonian.
Prepring.

[6] E. Ghaderpour and D. W. Morris.
Cayley graphs of order $30p$ are Hamiltonian.
Discrete Math., 312(24):3614–3625, 2012.

Bibliography III

[7] W. Holsztyński and R. F. E. Strube.
Paths and circuits in finite groups.
Discrete Math., 22(3):263–272, 1978.

[8] M. Kalkowski, M. Karoński, and F. Pfender.
Vertex coloring edge weightings with integer weights at most 6.
Rostock. Math. Kolloq., (64):39–43, 2009.

[9] M. Kalkowski, M. Karoński, and F. Pfender.
Vertex-coloring edge-weightings: towards the 1-2-3-conjecture.
J. Combin. Theory Ser. B, 100(3):347–349, 2010.

Bibliography IV

[10] M. Karoński, T. Ł uczak, and A. Thomason.
Edge weights and vertex colours.
J. Combin. Theory Ser. B, 91(1):151–157, 2004.

[11] I. Pak and R. Radoičić.
Hamiltonian paths in Cayley graphs.
Discrete Math., 309(17):5501–5508, 2009.

[12] J. Przybyło.
The 1-2-3 conjecture almost holds for regular graphs.
J. Combin. Theory Ser. B, 147:183–200, 2021.

[13] T. Wang and Q. Yu.
On vertex-coloring 13-edge-weighting.
Front. Math. China, 3(4):581–587, 2008.

Bibliography V

[14] D. Witte.
Cayley digraphs of prime-power order are Hamiltonian.
J. Combin. Theory Ser. B, 40(1):107–112, 1986.