What are... the Euler-Lagrange equations?

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What is...? Seminar

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Outline

1. Functionals in optimization
2. Stationary points
3. Deriving the Euler-Lagrange equations
4. Applications

Functionals in Optimization

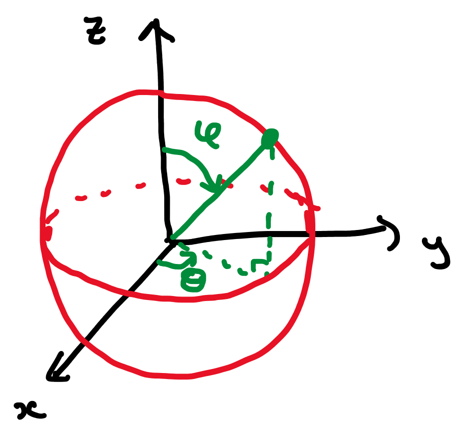
Geodesics

What is the path with the shortest length between two points?

In $\mathbb{R}^2$: minimize arclength $\ell(\alpha) := \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$

among curves $\alpha(t) = (x(t), y(t))$ with $\alpha(0) = P$ and $\alpha(1) = Q$.

In S^2: minimize $\ell(\alpha) := \int_0^1 \sqrt{\dot{\varphi}^2 + \sin^2\varphi\, \dot{\theta}^2}\, dt$



Brachistochrone problem

Along what curve will a ball roll from A to B in the shortest time, assuming a uniform gravitational field and no friction?

$$E = \frac{1}{2}mv^2 - mgy = 0$$

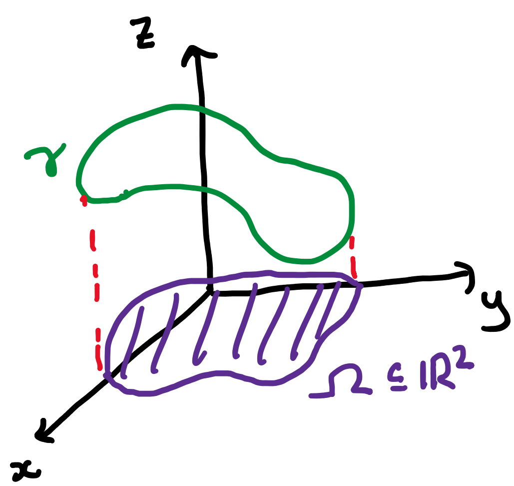
$$v = \sqrt{2gy}$$

Minimize $T(f) := \int_A^B \frac{ds}{v} = \int_0^{x_0} \frac{\sqrt{1+f'(x)^2}}{\sqrt{2gf(x)}}\,dx$

w/ $f(0)=0$, $f(x_0)=y_0$

Minimal surface

Given a boundary curve $\gamma: S^1 \to \mathbb{R}^3$, find the surface M with minimal surface area such that $\partial M = \gamma$.



Minimize $A(f) := \iint_\Omega (1+f_x^2+f_y^2)^{1/2}\,dx\,dy$

w/ $f(\partial\Omega)=\gamma$

Stationary Points

Fermat's Theorem: Let $f:(a,b)\to\mathbb{R}$ be a differentiable function. If f has a local extremum at $x_0\in(a,b)$, then $f'(x_0)=0$.

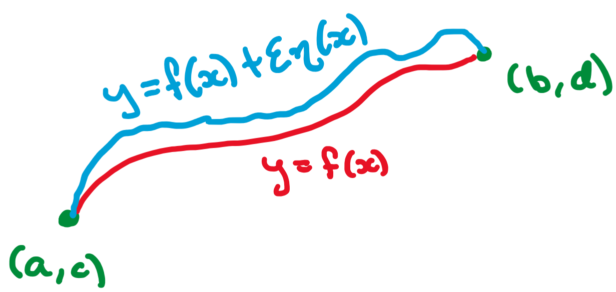
Expanding in series,

$$f(x_0+\varepsilon)=f(x_0)+\underbrace{\varepsilon f'(x_0)}_{\substack{\text{first order}\\ \text{change / first}\\ \text{variation}=0}}+\frac{\varepsilon^2}{2}f''(x_0)+O(\varepsilon^3)$$

(Weakly) stationary points for functionals

$$\underset{\text{action functional}}{S} := \int_a^b \overset{\text{Lagrangian}}{\mathcal{L}}(\underline{x},\underline{y},\underline{y'})\,dx \quad \text{for curves } y(x)$$

with $y(a)=c$ and $y(b)=d$. Assume $\mathcal{L}$ is smooth.



$\eta(a)=\eta(b)=0$

Using Taylor expansion:

$$S[f+\varepsilon\eta] = \int_a^b \mathcal{L}(x, f(x) + \varepsilon\eta(x), f'(x) + \varepsilon\eta'(x))\,dx$$

$$= S[f] + \varepsilon \underbrace{\int_a^b \left(\eta(x)\frac{\partial \mathcal{L}}{\partial y} + \eta'(x)\frac{\partial \mathcal{L}}{\partial y'}\right) dx}_{\text{first variation } \delta S} + O(\varepsilon^2)$$

A curve is stationary (for weak variations) if $\delta S = 0$ for every $\eta : [a,b] \to \mathbb{R}$ w/ $\eta(a) = \eta(b) = 0$.

Euler—Lagrange Equations

Goal: Maximize or minimize an action functional

$$S := \int_a^b \mathcal{L}(x, y, y')\, dx .$$

① A local extremum will occur at a stationary point ($\delta S = 0$).

② Integrate by parts:

$$\int_a^b \left(\eta(x) \frac{\partial \mathcal{L}}{\partial y}(x, f(x), f'(x)) + \eta'(x) \frac{\partial \mathcal{L}}{\partial y'}(x, f(x), f'(x)) \right) dx$$

$$= \int_a^b \eta \frac{\partial \mathcal{L}}{\partial y}\, dx + \cancelto{0}{\eta \frac{\partial \mathcal{L}}{\partial y'} \Big|_a^b} - \int_a^b \eta \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) dx$$

$$= \int_a^b \eta \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right)\right) dx$$

③ The curve η is arbitrary, so

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) = 0$$

(Euler–Lagrange eq.)

Multi-variable Euler–Lagrange equations: An action functional

$S = \int_a^b \mathcal{L}(t, x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n)\, dt$ is stationary for weak variations at $\vec{x} = \vec{r}(t)$ if and only if $\vec{r}$ satisfies the system of equations

$$\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = 0$$

for $i = 1, \ldots, n$.

Geodesics

$\mathbb{R}^2$

Minimize $\ell = \int^1 \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$

Lagrangian $\mathcal{L}(t, x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2}$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{\dot{x}}{v}$$

Euler–Lagrange equation:

$$\frac{d}{dt}\left(\frac{\dot{x}}{v}\right) = \frac{d}{dt}\left(\frac{\dot{y}}{v}\right) = 0.$$

$\dot{x} = cv$, $\dot{y} = dv \Rightarrow \frac{dy}{dx} = \text{const.}$

$\mathbb{S}^2$

Minimize $\ell = \int_0^1 \sqrt{\dot{\varphi}^2 + \sin^2\varphi\, \dot{\theta}^2}\, dt$

Assume $\theta = \theta(\varphi)$ so that $\ell = \int_{\varphi_0}^{\varphi_1} \sqrt{1 + \sin^2\varphi\, \theta'(\varphi)^2}\, d\varphi.$

Lagrangian $\mathcal{L}(\varphi, \theta, \theta') = \sqrt{1 + \sin^2\varphi\, (\theta')^2}$ does not depend on θ, so integrating the Euler–Lagrange equation wrt φ:

$$\frac{\partial \mathcal{L}}{\partial \theta'} = \frac{\sin^2 \varphi \, \theta'}{\sqrt{1+\sin^2\varphi (\theta')^2}} = \sin \alpha \qquad (*)$$

for some fixed α.

Solving $(*)$ for θ', we get

$$\frac{d\theta}{d\varphi} = \frac{\sin \alpha}{\sin \varphi \sqrt{\sin^2\varphi - \sin^2\alpha}} .$$

Substituting $\cos u = \frac{\tan \alpha}{\tan \varphi}$, we have $\frac{d\theta}{du} = 1$. Thus,

$$\cos(\theta + \beta) = \frac{\tan \alpha}{\tan \varphi}$$

In Cartesian coordinates,

$$x \cos \beta - y \sin \beta = z \tan \alpha .$$

This is a plane containing $(0,0,0)$, so geodesics are the intersection of $\mathbb{S}^2$ with a plane containing the origin (great circles).

Brachistochrone Problem

Minimize $T(f) := \int_A^B \frac{ds}{v} = \int_0^{x_0} \frac{\sqrt{1+f'(x)^2}}{\sqrt{2gf(x)}}\,dx$

Lagriangian: $\mathcal{L}(x,y,y') = \sqrt{\frac{1+(y')^2}{y}}$ does not involve x.

Now, multiplying the Euler–Lagrange equation by y', we have

$$\rightarrow \quad y'\frac{\partial \mathcal{L}}{\partial y} - y'\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) = 0.$$

But

$$\frac{d}{dx}\left(\mathcal{L}(x,y,y')\right) = \frac{\partial \mathcal{L}}{\partial y}y' + \frac{\partial \mathcal{L}}{\partial y'}y''$$

and

$$\frac{d}{dx}\left(y'\frac{\partial \mathcal{L}}{\partial y'}\right) = y''\frac{\partial \mathcal{L}}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right)$$

Thus,

$$\frac{d}{dx}\left(\mathcal{L} - y'\frac{\partial \mathcal{L}}{\partial y'}\right) = 0.$$

Theorem: Stationary points of an action functional $S = \int_a^b \mathcal{L}(y, y')\,dx$ are solutions of the equation

$$\boxed{\mathcal{L} - y' \frac{\partial \mathcal{L}}{\partial y'} = c}$$

Returning to the Brachistochrone problem, we want to solve

$$\sqrt{\frac{1+(y')^2}{y}} - y' \frac{y'}{\sqrt{y(1+(y')^2)}} = \text{const.}$$

Multiply by $\sqrt{y(1+(y')^2)}$:

$$1 + (y')^2 - (y')^2 = (\text{const.}) \sqrt{y(1+(y')^2)}$$

So,

$$y(1 + (y')^2) = c$$

Write $y'(x) = \tan \psi(x)$ so that $1 + (y')^2 = \sec^2 \psi$.

Then

$$y = c \cos^2 \psi$$

Hence,

$$\tan \psi = y' = -2c \cos \psi \sin \psi$$

$$2\cos^2 \psi \, d\psi = -\frac{1}{c} dx$$

Substituting $\varphi = 2\psi$ and integrating, we get

$$\begin{cases} x = a - b(\varphi + \sin \varphi) \\ y = b(1 + \cos \varphi) \end{cases}$$

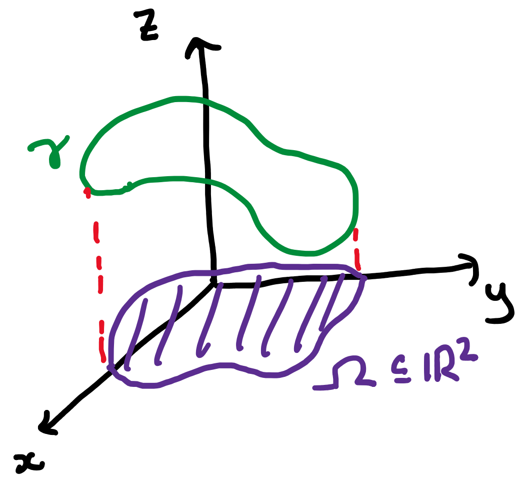
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} - \underbrace{b\begin{pmatrix} \varphi \\ -1 \end{pmatrix}} - \underbrace{b\begin{pmatrix} \sin \varphi \\ -\cos \varphi \end{pmatrix}}$$

This parametrizes a cycloid!

Observation: One can show that the time it takes to reach the lowest point on the curve is $\pi\sqrt{\frac{b}{g}}$, which is independent of the starting point!

The cycloid also solves the tautochrone/isochrone problem.

Minimal Surfaces



Minimize $A(f) := \iint_\Omega (1 + f_x^2 + f_y^2)^{1/2}\, dx\, dy$

Need Euler–Lagrange-type equation for $\mathcal{L}(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y})$.

$$S[f] = \iint_\Omega \mathcal{L}(x, y, f(x,y), f_x, f_y)\, dx\, dy \quad \text{with } f(\partial\Omega) = \gamma.$$

Using Taylor expansion,

$$S[f + \varepsilon\eta] = S[f] + \varepsilon \boxed{\iint_\Omega \left(\eta \frac{\partial \mathcal{L}}{\partial z} + \underline{\eta_x \frac{\partial \mathcal{L}}{\partial z_x} + \eta_y \frac{\partial \mathcal{L}}{\partial z_y}} \right) dx\, dy} + o(\varepsilon^2)$$

$$(\eta : \Omega \to \mathbb{R} \text{ satisfies } \eta|_{\partial\Omega} \equiv 0)$$

Now, by Stokes' theorem,

$$\iint_\Omega \left(\frac{\partial}{\partial x}\left(\eta \frac{\partial \mathcal{L}}{\partial z_x}\right) + \frac{\partial}{\partial y}\left(\eta \frac{\partial \mathcal{L}}{\partial z_y}\right) \right) = \overset{0}{\cancel{\iint_{\partial\Omega} \eta \left(\frac{\partial \mathcal{L}}{\partial z_x} dy - \frac{\partial \mathcal{L}}{\partial z_y} dx \right)}}$$

$$\underbrace{\left(\eta_x \frac{\partial \mathcal{L}}{\partial z_x} + \eta_y \frac{\partial \mathcal{L}}{\partial z_y}\right)} + \eta\left(\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial z_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial z_y}\right)\right)$$

Therefore,

$$\delta S = \iint_\Omega \eta\left(\frac{\partial \mathcal{L}}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial z_y}\right)\right) dx\, dy$$

Characteristic Equation:

$$\boxed{\frac{\partial \mathcal{L}}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial z_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial z_y}\right) = 0}$$

For $\mathcal{L} = (1 + z_x^2 + z_y^2)^{1/2}$, we get the equation

$$\underbrace{(1+f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1+f_x^2) f_{yy}}_{\text{related to mean curvature}} = 0$$

(Lagrange's equation)

Principle of Least (Stationary) Action

Newton's 2nd Law of Motion: $F = ma$.

As a diff. eq.:

$$-\frac{dV}{dx} = m\ddot{x}$$

This is equivalent to E–L eq. for

$$\mathcal{L}(t, x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - V(x) = \text{Kinetic} - \text{potential}$$

Principle of Least Action: Trajectories are stationary points for $S = \int \left(\frac{1}{2} m \dot{x}^2 - V(x)\right) dt$

Other Lagrangians:

- free-falling particle in general relativity:

$$\mathcal{L} = \frac{d\tau}{dt} \quad (\tau = \text{proper time})$$

- relativistic particle in E+M:

$$\mathcal{L} = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} - q(\phi - \vec{v} \cdot \vec{A})$$

- electric potential ϕ due to charge distribution ρ:

$$\mathcal{L} = \frac{\varepsilon_0}{2} |\nabla \phi|^2 - \rho \phi$$

(Gauss's law)

References

- Feynman Lectures on Physics, "The Principle of Least Action"
- Charles Fox. An Introduction to the Calculus of Variations.
- Heinrich W. Guggenheimer. Differential Geometry.