2022 Gordon examination problems

1. For a real number x, let $\lfloor x \rfloor$ denote the integer part of x. Prove that the sequence $\lfloor n\sqrt{2} \rfloor$, $n = 1, 2, 3, \ldots$, contains infnitely many powers of 2.

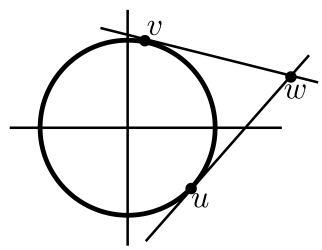
2. Recall that $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Prove that for all $x, y > 0$,

$$\left(1 - \sqrt{\tanh x}\right)\left(1 - \sqrt{\tanh y}\right) < 1 - \sqrt{\tanh(x+y)}.$$

3. Let (a_n) and (b_n) be two sequences of real numbers with $a_n, b_n \geq 1$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \frac{1}{n} \log a_n = a$ and $\lim_{n\to\infty} \frac{1}{n} \log b_n = b$. (Here log denotes the natural logarithm.) Prove:

$$\lim_{n\to\infty} \tfrac{1}{n} \log(a_n + b_n) = \max\{a, b\}.$$

4. Let u, v be two points in $\mathbb{C}$ lying on the circle $C = \{z : |z| = R\}$, and let the tangent lines to C at points u and v intersect at point w. Prove that $w = 2uv/(u+v)$.



5. Let A and B be orthogonal $n \times n$ matrices such that $\det A + \det B = 0$. Prove that $\det(A + B) = 0$.

6. Find a pair (n, m) of integers such that $n^2 + m^2 = 12101210$.