2022 Rasor-Bareis examination problems

1. Prove that for all positive integers n, $10^n - 1$ cannot be a perfect cube.

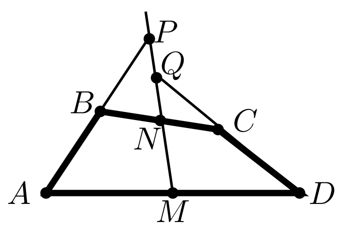
2. For a real number x, let $\{x\}$ denote the fractional part of x. Prove that $\left\{2^n\sqrt{2}\right\} > 1/2$ for infinitely many $n \in \mathbb{N}$.

3. Let A be a positive real number. If (a_n) is a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} a_n = A$, find all possible values the sum $\sum_{n=1}^{\infty} a_n^2$ could have.

4. Prove that

$$\int_0^{\pi/2} \frac{(\cos x)^{\sin x} dx}{(\cos x)^{\sin x} + (\sin x)^{\cos x}} = \frac{\pi}{4}.$$

5. Let $ABCD$ be a convex quadrilateral satisfying $|AB| = |CD|$. Let M and N be the midpoints of AD and BC respectively, and let the ray MN intersect the rays AB and DC at points P and Q respectively. Show that the angles $\angle APM$ and $\angle DQM$ are equal.



6. Assume that three faces of a tetrahedron are pairwise orthogonal and have areas a, b, c, and let the area of the fourth face be d. Prove that $d^2 = a^2 + b^2 + c^2$.