2022 Gordon exam solutions

1. For a real number x, let $\lfloor x \rfloor$ denote the integer part of x. Prove that the sequence $\lfloor n\sqrt{2} \rfloor$, $n = 1, 2, 3, \ldots$, contains infnitely many powers of 2.

Solution. To have $\lfloor n\sqrt{2} \rfloor = 2^k$ for $k \in \mathbb{N}$ we need $2^k < n\sqrt{2} < 2^k + 1$, that is, $2^{k-1}\sqrt{2} < n < 2^{k-1}\sqrt{2} + \frac{1}{\sqrt{2}}$. Such n exists if $\{2^{k-1}\sqrt{2}\} > 1 - \frac{1}{\sqrt{2}}$ (where $\{x\}$ stands for the fractional part of x), and it suffices if $\{2^{k-1}\sqrt{2}\} > 1/2$. So, it is enough to prove that $\{2^k\sqrt{2}\} > 1/2$ for infinitely many $k \in \mathbb{N}$.

Since $\sqrt{2}$ is irrational, $k\sqrt{2}$ is irrational for all integer k, so $\{2^n\sqrt{2}\} \neq 0$ for all n. Therefore for every $m \in \mathbb{N}$ we have $(1/2)^{d+1} < \{2^m\sqrt{2}\} < (1/2)^d$ for some integer $d \geq 0$; so $1/2 < 2^d\{2^m\sqrt{2}\} < 1$, and so $\{2^{m+d}\sqrt{2}\} = 2^d\{2^m\sqrt{2}\} > 1/2$. We obtain that for every $m \in \mathbb{N}$ there exists $n \geq m$ such that $\{2^n\sqrt{2}\} > 1/2$; this proves that the set of such n is infinite.

2. Recall that $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Prove that for all $x, y > 0$,

$$\left(1 - \sqrt{\tanh x}\right)\left(1 - \sqrt{\tanh y}\right) < 1 - \sqrt{\tanh(x + y)}.$$

Solution. We use the (well known) equality $\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$. Putting $u = \sqrt{\tanh x}$ and $v = \sqrt{\tanh y}$, we need to prove that $(1 - u)(1 - v) < 1 - \sqrt{\frac{u^2+v^2}{1+u^2v^2}}$, which is equivalent to

$$\frac{u^2 + v^2}{1 + u^2v^2} < \big(1 - (1 - u)(1 - v)\big)^2,$$

with $0 < u, v < 1$.

Since $(1 - u)(1 - v) > 0$ we have $1 + uv > u + v$, so $2(1 + uv) > 1 + uv + u + v = (1 + u)(1 + v)$, so $2 > \frac{(1+u)(1+v)}{1+uv}$, and

$$2uv > \frac{(1 + u)(1 + v)uv}{1 + uv} = \frac{(1 + u)(1 + v)\big((1 + uv) - 1\big)}{1 + uv} = (1 + u)(1 + v) - \frac{(1 + u)(1 + v)}{1 + uv}.$$

So,

$$\frac{(1 + u)(1 + v)}{1 + u^2v^2} > \frac{(1 + u)(1 + v)}{1 + uv} > (1 + u)(1 + v) - 2uv = 2 - (1 - u)(1 - v).$$

Hence,

$$\frac{(1 - u^2)(1 - v^2)}{1 + u^2v^2} > 2(1 - u)(1 - v) - (1 - u)^2(1 - v)^2,$$

and

$$\frac{u^2 + v^2}{1 + u^2v^2} = 1 - \frac{(1 - u^2)(1 - v^2)}{1 + u^2v^2} < 1 - 2(1 - u)(1 - v) + (1 - u)^2(1 - v)^2 = \big(1 - (1 - u)(1 - v)\big)^2.$$

3. Let (a_n) and (b_n) be two sequences of real numbers with $a_n, b_n \geq 1$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \frac{1}{n} \log a_n = a$ and $\lim_{n\to\infty} \frac{1}{n} \log b_n = b$. (Here log denotes the natural logarithm.) Prove:

$$\lim_{n\to\infty} \tfrac{1}{n} \log(a_n + b_n) = \max\{a, b\}.$$

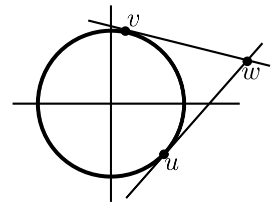
Solution. Let $a \neq b$; without loss of generality assume that $a < b$. Then $a_n \leq b_n$ for all n large enough. Write

$$\tfrac{1}{n}\log(a_n + b_n) = \tfrac{1}{n}\log b_n + \tfrac{1}{n}\log(1 + a_n/b_n) = \tfrac{1}{n}\log b_n + \log(1 + a_n/b_n)^{1/n}, \quad n \in \mathbb{N}.$$

Since $1 + a_n/b_n \leq 2$ for large n, we have $(1 + a_n/b_n)^{1/n} \longrightarrow 1$ as $n \longrightarrow \infty$, thus $\log(1 + a_n/b_n)^{1/n} \longrightarrow 0$, and so $\frac{1}{n}\log(a_n + b_n) \longrightarrow b$.

Now let $a = b$. Then we also have $\lim_{n\to\infty} \frac{1}{n} \log 2a_n = \lim_{n\to\infty} \frac{1}{n} \log a_n + \lim_{n\to\infty} \frac{1}{n} \log 2 = a$ and $\lim_{n\to\infty} \frac{1}{n} \log 2b_n = a$. Since for every n, $2a_n \leq a_n + b_n \leq 2b_n$ or $2b_n \leq a_n + b_n \leq 2a_n$, by the squeeze theorem, $\lim_{n\to\infty} \frac{1}{n} \log(a_n + b_n) = a$.

4. Let u, v be two points in $\mathbb{C}$ lying on the circle $C = \{z : |z| = R\}$, and let the tangent lines to C at points u and v intersect at point w. Prove that $w = 2uv/(u+v)$.



Solution. Let $z = 2uv/(u+v)$; if we show that the vector $z-u$ is orthogonal to u and $z-v$ is orthogonal to v, this will imply that the lines $\overline{uz}$ and $\overline{vz}$ are tangent to C, and so, $z = w$. We have $z-u = (uv-u^2)/(u+v)$ and so

$$\frac{z-u}{u} = \frac{v-u}{u+v} = \frac{(v-u)(\overline{u}+\overline{v})}{|u+v|^2} = \frac{|v|^2-|u|^2+v\overline{u}-u\overline{v}}{|u+v|^2} = \frac{v\overline{u}-u\overline{v}}{|u+v|^2} = \frac{v\overline{u}-\overline{v\overline{u}}}{|u+v|^2} \in i\mathbb{R};$$

so $z-u = icu$ for some $c \in \mathbb{R}$, so $z-u \perp u$. The same way, $z-v \perp v$.

5. Let A and B be orthogonal $n \times n$ matrices such that $\det A + \det B = 0$. Prove that $\det(A+B) = 0$.

Solution. Orthogonal matrices have determinant ± 1, thus assume that $\det A = 1$ and $\det B = -1$. Then

$$\det(A+B) = -\det A^T \det(A+B) \det B^T = -\det\bigl(A^T A B^T + A^T B B^T\bigr) = -\det(B^T + A^T) = -\det(A+B).$$

Hence, $\det(A+B) = 0$.

Another solution. Complex non-real eigenvalues of orthogonal matrices appear in pairs: α, $\overline{\alpha}$, and real eigenvalues are equal to ± 1. After multiplying (from the left) both A and B by A^{-1}, we may assume that $A = I$. Then B is an orthogonal matrix with $\det B = -1$. It follows that B has a real eigenvalue equal to -1, and if u is the corresponding vector, then $(I+B)u = 0$. So, $I+B$ is degenerate, and has zero determinant.

6. Find a pair (n, m) of integers such that $n^2 + m^2 = 12101210$.

Solution. We have $12101210 = 121 \cdot 10 \cdot 10001 = 11^2 \cdot (3^2+1^2) \cdot (100^2+1^2)$. Using the nice identity $(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2$, we get $12101210 = 11^2 \cdot (301^2 + 97^2) = 3311^2 + 1067^2$.

Another solution. There is, actually, a criterion for an integer to be representable as a sum of two squares, and an algorithm that allows to find (all) such representations. The prime factorization of 12101210 is $2 \cdot 5 \cdot 11^2 \cdot 73 \cdot 137$. The only prime factor here that is congruent to 3 modulo 4 is 11, and it appears with an even exponent; hence 12101210 is representable as a sum of two squares. To find such a representation we write $2 = (1+i)(1-i)$, $5 = (2+i)(2-i)$, $73 = (8+3i)(8-3i)$, and $137 = (11+4i)(11-4i)$; then compute, for instance, $(1+i)(2+i)11(8+3i)(11+4i) = -1309 + 3223i$, and get $12101210 = 1309^2 + 3223^2$.