2022 Gordon exam solutions

1. For a real number $x$, let $\lfloor x \rfloor$ denote the integer part of $x$. Prove that the sequence $\lfloor n \sqrt{2} \rfloor$, $n = 1, 2, 3, \ldots$, contains infinitely many powers of 2.

**Solution.** To have $\lfloor n \sqrt{2} \rfloor = 2^k$ for $k \in \mathbb{N}$ we need $2^k < n \sqrt{2} < 2^k + 1$, that is, $2^{k-1} \sqrt{2} < n < 2^{k-1} \sqrt{2} + \frac{1}{\sqrt{2}}$. Such $n$ exists if $\{2^{k-1} \sqrt{2}\} > 1 - \frac{1}{\sqrt{2}}$ (where $\{x\}$ stands for the fractional part of $x$), and it suffices if $\{2^{k-1} \sqrt{2}\} > 1/2$. So, it is enough to prove that $\{2^{k} \sqrt{2}\} > 1/2$ for infinitely many $k \in \mathbb{N}$.

Since $\sqrt{2}$ is irrational, $k \sqrt{2}$ is irrational for all integer $k$, so $\{2^{m} \sqrt{2}\} \neq 0$ for all $n$. Therefore for every $m \in \mathbb{N}$ we have $(1/2)^{d+1} \{2^{m} \sqrt{2}\} < (1/2)^{d}$ for some integer $d \geq 0$; so $1/2 < 2^d\{2^{m} \sqrt{2}\} < 1$, and so $\{2^{m+d} \sqrt{2}\} = 2^d \{2^{m} \sqrt{2}\} > 1/2$. We obtain that for every $m \in \mathbb{N}$ there exists $n \geq m$ such that $\{2^{n} \sqrt{2}\} > 1/2$; this proves that the set of such $n$ is infinite.

2. Recall that $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. Prove that for all $x, y > 0$,

$$(1 - \sqrt{\tanh x})(1 - \sqrt{\tanh y}) < 1 - \sqrt{\tanh(x + y)}.$$ 

**Solution.** We use the (well known) equality $\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$. Putting $u = \sqrt{\tanh x}$ and $v = \sqrt{\tanh y}$, we need to prove that $(1 - u)(1 - v) < 1 - \sqrt{\frac{u^2 + v^2}{1 + uv}}$, which is equivalent to

$$\frac{u^2 + v^2}{1 + uv} > (1 - u)(1 - v)\left(1 - \frac{1}{1 - u)(1 - v)}\right)^2,$$

with $0 < u, v < 1$.

Since $(1 - u)(1 - v) > 0$ we have $1 + uv > u + v$, so $2(1 + uv) > 1 + uv + u + v = (1 + u)(1 + v)$, so

$$2 > \frac{(1+u)(1+v)}{1+uv},$$

and

$$2uv > \frac{(1+u)(1+v)uv}{1+uv} = \frac{(1+u)(1+v)((1+uv)-1)}{1+uv} = (1+u)(1+v) - (1+u)(1+v).$$

So,

$$\frac{(1+u)(1+v)}{1+uv} > \frac{(1+u)(1+v)}{1+uv} > (1+u)(1+v) - 2uv = 2 - (1-u)(1-v).$$

Hence,

$$\frac{(1-u^2)(1-v^2)}{1+u^2v^2} > 2(1-u)(1-v) - (1-u)^2(1-v)^2,$$

and

$$\frac{u^2 + v^2}{1 + uv} = 1 - \frac{(1-u^2)(1-v^2)}{1+u^2v^2} < 1 - 2(1-u)(1-v) + (1-u)^2(1-v)^2 = (1 - (1-u)(1-v))^2.$$

3. Let $(a_n)$ and $(b_n)$ be two sequences of real numbers with $a_n, b_n \geq 1$ for all $n \in \mathbb{N}$ such that $\lim_{n \to \infty} \frac{1}{n} \log a_n = a$ and $\lim_{n \to \infty} \frac{1}{n} \log b_n = b$. (Here log denotes the natural logarithm.) Prove:

$$\lim_{n \to \infty} \frac{1}{n} \log (a_n + b_n) = \max\{a, b\}.$$

**Solution.** Let $a \neq b$; without loss of generality assume that $a < b$. Then $a_n \leq b_n$ for all $n$ large enough. Write

$$\frac{1}{n} \log (a_n + b_n) = \frac{1}{n} \log b_n + \frac{1}{n} \log (1 + a_n/b_n) = \frac{1}{n} \log b_n + \log (1 + a_n/b_n)^{1/n}, \quad n \in \mathbb{N}.$$

Since $1 + a_n/b_n \leq 2$ for large $n$, we have $(1 + a_n/b_n)^{1/n} \to 1$ as $n \to \infty$, thus $\log (1 + a_n/b_n)^{1/n} \to 0$, and so $\frac{1}{n} \log (a_n + b_n) \to b$.

Now let $a = b$. Then we also have $\lim_{n \to \infty} \frac{1}{n} \log 2a_n = \lim_{n \to \infty} \frac{1}{n} \log a_n + \lim_{n \to \infty} \frac{1}{n} \log 2 = a$ and $\lim_{n \to \infty} \frac{1}{n} \log 2b_n = a$. Since for every $n$, $2a_n \leq a_n + b_n \leq 2b_n$ or $2b_n \leq a_n + b_n \leq 2a_n$, by the squeeze theorem, $\lim_{n \to \infty} \frac{1}{n} \log (a_n + b_n) = a$. 

1
4. Let \( u, v \) be two points in \( \mathbb{C} \) lying on the circle \( C = \{ z : |z| = R \} \), and let the tangent lines to \( C \) at points \( u \) and \( v \) intersect at point \( w \). Prove that \( w = 2uv/(u+v) \).

\[ \text{Solution.} \quad \text{Let } z = 2uv/(u+v); \text{ if we show that the vector } z - u \text{ is orthogonal to } u \text{ and } z - v \text{ is orthogonal to } v, \text{ this will imply that the lines } u\pi \text{ and } v\pi \text{ are tangent to } C, \text{ and so, } z = w. \text{ We have } z - u = (uv - u^2)/(u+v) \text{ and so } \]

\[ \frac{z-u}{u} = \frac{v-u}{u+v} = \frac{(v-u)(\pi + \pi)}{|u+v|^2} = \frac{|v|^2 - |u|^2}{|u+v|^2} = \frac{v\pi - u\pi}{|u+v|^2} \in i\mathbb{R}; \]

so \( z - u = icu \) for some \( c \in \mathbb{R} \), so \( z - u \perp u \). The same way, \( z - v \perp v \).

5. Let \( A \) and \( B \) be orthogonal \( n \times n \) matrices such that \( \det A + \det B = 0 \). Prove that \( \det(A + B) = 0 \).

\[ \text{Solution.} \quad \text{Orthogonal matrices have determinant } \pm 1, \text{ thus assume that } \det A = 1 \text{ and } \det B = -1. \text{ Then } \]

\[ \det(A + B) = -\det A \det(A + B) \det B^T = -\det(A^TABB^T + A^TBB^T) = -\det(B^T + A^T) = -\det(A + B). \]

Hence, \( \det(A + B) = 0 \).

\[ \text{Another solution.} \quad \text{Complex non-real eigenvalues of orthogonal matrices appear in pairs: } \alpha, \alpha^*, \text{ and real eigenvalues are equal to } \pm 1. \text{ After multiplying (from the left) both } A \text{ and } B \text{ by } A^{-1}, \text{ we may assume that } A = I. \text{ Then } B \text{ is an orthogonal matrix with } \det B = -1. \text{ It follows that } B \text{ has a real eigenvalue equal to } -1, \text{ and if } u \text{ is the corresponding vector, then } (I + B)u = 0. \text{ So, } I + B \text{ is degenerate, and has zero determinant.} \]

6. Find a pair \( (n, m) \) of integers such that \( n^2 + m^2 = 12101210 \).

\[ \text{Solution.} \quad \text{We have } 12101210 = 121 \cdot 10 \cdot 10001 = 11^2 \cdot (3^2 + 1^2) \cdot (100^2 + 1^2). \text{ Using the nice identity } (a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2, \text{ we get } 12101210 = 11^2 \cdot (301^2 + 97^2) = 3311^2 + 1067^2. \]

\[ \text{Another solution.} \quad \text{There is, actually, a criterion for an integer to be representable as a sum of two squares, and an algorithm that allows to find (all) such representations. The prime factorization of } 12101210 \text{ is } 2 \cdot 5 \cdot 11^2 \cdot 73 \cdot 137. \text{ The only prime factor here that is congruent to 3 modulo 4 is } 11, \text{ and it appears with an even exponent; hence } 12101210 \text{ is representable as a sum of two squares. To find such a representation we write } 2 = (1 + i)(1 - i), 5 = (2 + i)(2 - i), 73 = (8 + 3i)(8 - 3i), \text{ and } 137 = (11 + 4i)(11 - 4i); \text{ then compute, for instance, } (1 + i)(2 + i)11(8 + 3i)(11 + 4i) = -1309 + 3223i, \text{ and get } 12101210 = 1309^2 + 3223^2. \]