2022 Rasor-Bareis exam solutions

1. Prove that for all positive integers n, $10^n - 1$ cannot be a perfect cube.

Solution. By the way of contradiction, assume that $10^n - 1$ is a perfect cube for some n. Then, since $10^n - 1$ is divisible by 9, it must be divisible by 27, so $(10^n - 1)/9 = \underbrace{11\cdots 11}_{n}$ is divisible by 3, so n is divisible by 3, $n = 3k$. But then both $10^n - 1$ and $10^n = 10^{3k}$ are perfect cubes, which cannot be true.

Another solution. By the way of contradiction, assume that $10^n - 1 = N^3$ for some integers n and N. Then $10^n = N^3 + 1 = (N+1)(N^2 - N + 1)$. It follows that the prime divisors of both $N + 1$ and $N^2 - N + 1$ can only be 2 and 5, $N + 1 = 2^k 5^l$ and $N^2 - N + 1 = 2^r 5^s$ for some k, l, r, s. Since N^3 is odd, N is odd, so $N + 1$ is even, and $N^2 - N + 1$ is odd. On the other hand, if $N + 1$ is divisible by 5, then $N \equiv -1 \bmod 5$, so $N^2 - N + 1 \equiv 3 \bmod 5$, so $N^2 - N + 1$ is not divisible by 5. Hence, the only option is that $N + 1 = 2^n$ and $N^2 - N + 1 = 5^n$. But then $4^n = (2^n)^2 = (N+1)^2 = N^2 + 2N + 1 > N^2 - N + 1 = 5^n$, contradiction.

2. For a real number x, let $\{x\}$ denote the fractional part of x. Prove that $\{2^n\sqrt{2}\} > 1/2$ for infinitely many $n \in \mathbb{N}$.

Solution. Since $\sqrt{2}$ is irrational, $k\sqrt{2}$ is irrational for all integer k, so $\{2^n\sqrt{2}\} \neq 0$ for all n. Therefore for every $m \in \mathbb{N}$ we have $(1/2)^{d+1} < \{2^m\sqrt{2}\} < (1/2)^d$ for some integer $d \geq 0$; so $1/2 < 2^d\{2^m\sqrt{2}\} < 1$, and so $\{2^{m+d}\sqrt{2}\} = 2^d\{2^m\sqrt{2}\} > 1/2$. We obtain that for every $m \in \mathbb{N}$ there exists $n \geq m$ such that $\{2^n\sqrt{2}\} > 1/2$; this proves that the set of such n is infinite.

Another solution. Let $1.c_1c_2c_3\ldots$, with $c_i \in \{0, 1\}$ for all i, be the binary expansion of $\sqrt{2}$. Then for every n, the binary expansion of $\{2^n\sqrt{2}\}$ is $0.c_{n+1}c_{n+2}c_{n+3}\ldots$, and we have $\{2^n\sqrt{2}\} > 1/2$ iff $c_{n+1} = 1$. (Notice that $\{2^n\sqrt{2}\} \neq 1/2$ for all n.) Since $\sqrt{2}$ is irrational, the sequence (c_i) contains infnitely many 1-s, so $\{2^n\sqrt{2}\} > 1/2$ for infnitely many n.

3. Let A be a positive real number. If (a_n) is a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} a_n = A$, find all possible values the sum $\sum_{n=1}^{\infty} a_n^2$ could have.

Solution. Let S be the set of possible values of the sum $\sum_{n=1}^{\infty} a_n^2$. First of all, if (a_n) is a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} a_n = A$, at least one of a_n must be positive, so $\sum_{n=1}^{\infty} a_n^2 > 0$. On the other hand, $a_n/A \leq 1$ for all n, so $a_n^2/A^2 \leq a_n/A$, so $\sum_{n=1}^{\infty} a_n^2/A^2 \leq \sum_{n=1}^{\infty} a_n/A = 1$, so $\sum_{n=1}^{\infty} a_n^2 \leq A^2$ (and for the sequence $(a_n) = (A, 0, 0, \ldots)$ we have $\sum_{n=1}^{\infty} a_n^2 = A^2$). Hence, $S \subseteq (0, A^2]$; we claim that $S = (0, A^2]$.

For any $m \in \mathbb{N}$ let $(a_{m,n})$ be the sequence $\left(\frac{A}{m}, \ldots, \frac{A}{m}, 0, 0\ldots\right)$, with m nonzero terms; then $\sum_{n=1}^{\infty} a_{m,n} = A$ and has $\sum_{n=1}^{\infty} a_{m,n}^2 = \frac{A^2}{m^2}m = \frac{A^2}{m}$. Fix $m \in \mathbb{N}$. For any $t \in [0, 1]$ the sequence $(b_{t,n}) = (1-t)(a_{m,n}) + t(a_{m+1,n}) = \big((1-t)a_{m,n} + ta_{m+1,n}\big)$ also satisfies $\sum_{n=1}^{\infty} b_{t,n} = A$, and has the property that $(b_{0,n}) = (a_{m,n})$ and $(b_{1,n}) = (a_{m+1,n})$. The function $f(t) = \sum_{n=1}^{\infty} b_{t,n}^2 = \sum_{n=1}^{m+1} b_{t,n}^2$ is continuous with $f(0) = \frac{A^2}{m}$ and $f(1) = \frac{A^2}{m+1}$. By the intermediate value theorem, f takes all values in the interval $\left[\frac{A^2}{m+1}, \frac{A^2}{m}\right]$. Hence, $\left[\frac{A^2}{m+1}, \frac{A^2}{m}\right] \subseteq S$ for all $m \in \mathbb{N}$; since the union of these intervals is $(0, A^2]$, we get that $(0, A^2] \subseteq S$.

4. Prove that

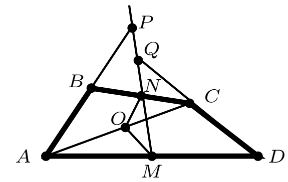
$$\int_0^{\pi/2} \frac{(\cos x)^{\sin x} dx}{(\cos x)^{\sin x} + (\sin x)^{\cos x}} = \frac{\pi}{4}.$$

Solution. Let $I = \int_0^{\pi/2} \frac{(\cos x)^{\sin x} dx}{(\cos x)^{\sin x} + (\sin x)^{\cos x}}$. After the substitution $y = \pi/2 - x$, we see that also $I = \int_0^{\pi/2} \frac{(\sin y)^{\sin y} dy}{(\sin y)^{\cos y} + (\cos y)^{\sin y}}$. Then

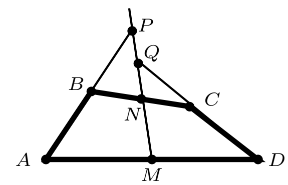
$$I + I = \int_0^{\pi/2} \frac{(\cos x)^{\sin x} dx}{(\cos x)^{\sin x} + (\sin x)^{\cos x}} + \int_0^{\pi/2} \frac{(\sin x)^{\cos x} dx}{(\sin x)^{\cos x} + (\cos x)^{\sin x}} = \int_0^{\pi/2} \frac{(\cos x)^{\sin x} + (\sin x)^{\cos x}}{(\cos x)^{\sin x} + (\sin x)^{\cos x}}\, dx$$
$$= \int_0^{\pi/2} dx = \pi/2.$$

Hence, $I = \pi/4$.

5. Let $ABCD$ be a convex quadrilateral satisfying $|AB| = |CD|$. Let M and N be the midpoints of AD and BC respectively, and let the ray MN intersect the rays AB and DC at points P and Q respectively. Show that the angles $\angle APM$ and $\angle DQM$ are equal.

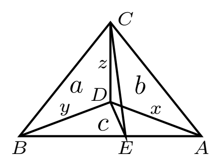


Solution. Let O be the midpoint of the interval AC. Then ON is a midline of $\triangle ACB$, so is parallel to AB, so $\angle ONM = \angle APM$, and OM is a midline of $\triangle CAD$, so is parallel to CD, so $\angle OMN = \angle DQM$. Also, $|ON| = \frac{1}{2}|AB| = \frac{1}{2}|CD| = |OM|$, so the triangle $\triangle NOM$ is isosceles, and $\angle OMN = \angle OMN$. Hence, $\angle APM = \angle DQN$.



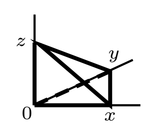
Another solution. By the sine theorem for triangles, $\frac{\sin\angle CQN}{|CN|} = \frac{\sin\angle CNQ}{|CQ|}$ and $\frac{\sin\angle DQM}{|DM|} = \frac{\sin\angle DMQ}{|DQ|}$, so $\frac{|DM|}{|CN|} = \frac{|DQ|}{|CQ|}\frac{\sin\angle CNQ}{\sin\angle DMQ}$. Similarly, $\frac{|AM|}{|BN|} = \frac{|AP|}{|BP|}\frac{\sin\angle BNP}{\sin\angle AMP}$. Since $|DM| = |AM|$, $|CN| = |BN|$, $\sin\angle CNQ = \sin\angle BNP$ and $\sin\angle DMQ = \sin\angle AMP$, we get that $\frac{|DQ|}{|CQ|} = \frac{|AP|}{|BP|}$. Since $|DC| = |AB|$, this implies that $|DQ| = |AP|$. So, $\sin\angle DQM = |DM|\frac{\sin\angle DMQ}{|DQ|} = |AM|\frac{\sin\angle AMQ}{|AP|} = \sin\angle APM$. Also, we have $\angle DMQ \geq \pi/2$ or $\angle AMP \geq \pi/2$; on either case $|DQ| = |AP| > |DM| = |AM|$, so both $\angle APM$ and $\angle DQM$ are acute. Hence, $\angle APM \simeq \angle DQM$.

6. Assume that three faces of a tetrahedron are pairwise orthogonal and have areas a, b, c, and let the area of the fourth face be d. Prove that $d^2 = a^2 + b^2 + c^2$.



Solution. Let A, B, C, D be the vertices of the tetrahedron opposite to the faces whose areas are a, b, c, d respectively, let $x = |DA|$, $y = |DB|$, $z = |DC|$, so that $a = \frac{1}{2}yz$, $b = \frac{1}{2}xz$, $c = \frac{1}{2}xy$. Let E be the point of intersection of the segment AB with the plane containing DC and orthogonal to AB.
Then $|AB| = \sqrt{x^2+y^2}$, $|DE| = xy/|AB|$, $|CE| = \sqrt{|DE|^2 + z^2}$, so

$$d^2 = \tfrac{1}{4}|AB|^2|CE|^2 = \tfrac{1}{4}(x^2+y^2)\big(x^2y^2/(x^2+y^2) + z^2\big) = \tfrac{1}{4}\big(x^2y^2 + x^2z^2 + y^2z^2\big) = c^2 + b^2 + a^2.$$



Another solution. Introduce the coordinate system such that the three orthogonal faces of the tetrahedron lay in the coordinate planes (and the common vertex of these faces is the origin of the coordinate system):
In this coordinate system, the other three vertices of the tetrahedron have coordinates $(x,0,0)$, $(0,y,0)$, and $(0,0,z)$, so that $a = \frac{xy}{2}$, $b = \frac{yz}{2}$, $c = \frac{yz}{2}$. The "area vector" of the fourth face is $u = \frac{1}{2}(-x,y,0)\times(-x,0,z) = \frac{1}{2}(yz,xz,xy)$, thus $d^2 = |u|^2 = \frac{1}{4}\big((yz)^2 + (xz)^2 + (xy)^2\big) = a^2 + b^2 + c^2$.