1. Prove that for all positive integers \( n \), \( 10^n - 1 \) cannot be a perfect cube.

Solution. By the way of contradiction, assume that \( 10^n - 1 \) is a perfect cube for some \( n \). Then, since \( 10^n - 1 \) is divisible by 9, it must be divisible by 27, so \( (10^n - 1)/9 = 11 \cdots \frac{n}{3} \) is divisible by 3, so \( n \) is divisible by 3, \( n = 3k \). But then both \( 10^n - 1 \) and \( 10^n = 10^{3k} \) are perfect cubes, which cannot be true.

Another solution. By the way of contradiction, assume that \( 10^n - 1 = N^3 \) for some integers \( n \) and \( N \). Then \( 10^n = N^3 + 1 = (N + 1)(N^2 - N + 1) \). It follows that the prime divisors of both \( N + 1 \) and \( N^2 - N + 1 \) can only be 2 and 5, \( N + 1 = 2^k 5^l \) and \( N^2 - N + 1 = 2^m 5^n \) for some \( k, l, r, s \). Since \( N^3 \) is odd, \( N \) is odd, so \( N + 1 \) is even, and \( N^2 - N + 1 \) is odd. On the other hand, if \( N + 1 \) is divisible by 5, then \( N = -1 \mod 5 \), so \( N^2 - N + 1 \) is odd and \( N^2 - N + 1 = 3 \mod 5 \), so \( N^2 - N + 1 \) is not divisible by 5. Hence, the only option is that \( N + 1 = 2^n \) and \( N^2 - N + 1 = 5^n \). But then \( 4^n = (2^n)^2 = (N + 1)^2 = N^2 + 2N + 1 > N^2 - N + 1 = 5^n \), contradiction.

2. For a real number \( x \), let \( \{ x \} \) denote the fractional part of \( x \). Prove that \( \{2^n \sqrt{2} \} > 1/2 \) for infinitely many \( n \in \mathbb{N} \).

Solution. Since \( \sqrt{2} \) is irrational, \( k \sqrt{2} \) is irrational for all integer \( k \), so \( \{2^n \sqrt{2} \} \neq 0 \) for all \( n \). Therefore for every \( m \in \mathbb{N} \) we have \( (1/2)^{d+1} < \{2^n \sqrt{2} \} < (1/2)^d \) for some integer \( d \geq 0 \); so \( 1/2 < 2^d \{2^n \sqrt{2} \} < 1 \), and so \( \{2^{n+d} \sqrt{2} \} = 2^d \{2^n \sqrt{2} \} > 1/2 \). We obtain that for every \( m \in \mathbb{N} \) there exists \( n \geq m \) such that \( \{2^n \sqrt{2} \} > 1/2 \); this proves that the set of such \( n \) is infinite.

Another solution. Let \( 1.c_1 c_2 c_3 \ldots \) with \( c_i \in \{0, 1\} \) for all \( i \), be the binary expansion of \( \sqrt{2} \). Then for every \( n \), the binary expansion of \( \{2^n \sqrt{2} \} \) is \( 0.c_{n+1} c_{n+2} c_{n+3} \ldots \), and we have \( \{2^n \sqrt{2} \} > 1/2 \) if \( c_{n+1} = 1 \). (Notice that \( \{2^n \sqrt{2} \} \neq 1/2 \) for all \( n \)). Since \( \sqrt{2} \) is irrational, the sequence \( (c_i) \) contains infinitely many 1-s, so \( \{2^n \sqrt{2} \} > 1/2 \) for infinitely many \( n \).

3. Let \( A \) be a positive real number. If \( (a_n) \) is a sequence of nonnegative real numbers with \( \sum_{n=1}^{\infty} a_n = A \), find all possible values the sum \( \sum_{n=1}^{\infty} a_n^2 \) could have.

Solution. Let \( S \) be the set of possible values of the sum \( \sum_{n=1}^{\infty} a_n^2 \). First of all, if \( (a_n) \) is a sequence of nonnegative real numbers with \( \sum_{n=1}^{\infty} a_n = A \), at least one of \( a_n \) must be positive, so \( \sum_{n=1}^{\infty} a_n^2 > 0 \). On the other hand, \( a_n/A \leq 1 \) for all \( n \), so \( a_n^2/A^2 \leq a_n/A \), so \( \sum_{n=1}^{\infty} a_n^2/A^2 \leq \sum_{n=1}^{\infty} a_n/A = 1 \), so \( \sum_{n=1}^{\infty} a_n^2 \leq A^2 \) (and for the sequence \( (a_n) = (A, 0, 0, \ldots) \) we have \( \sum_{n=1}^{\infty} a_n^2 = A^2 \)). Hence, \( S \subseteq (0, A^2] \); we claim that \( S = (0, A^2] \).

For any \( m \in \mathbb{N} \) let \( (a_{m,n}) \) be the sequence \( (\frac{1}{m}, \ldots, \frac{1}{m}, 0, 0, \ldots) \), with \( m \) nonzero terms; then \( \sum_{n=1}^{\infty} a_{m,n} = A \) and has \( \sum_{n=1}^{\infty} a_{m,n}^2 = \frac{A^2}{m^2} m = \frac{A^2}{m} \). Fix \( m \in \mathbb{N} \). For any \( t \in [0, 1] \) the sequence \( (b_{m,n}) = (1 - t)(a_{m,n}) + t(a_{m+1,n}) = (1 - t)a_{m,n} + ta_{m+1,n}) \) also satisfies \( \sum_{n=1}^{\infty} b_{m,n} = A \), and has the property that \( (b_{m,n}) = (a_{m,n}) \) and \( (b_{m+1,n}) = (a_{m+1,n}) \). The function \( f(t) = \sum_{n=1}^{\infty} b_{t,n}^2 = \sum_{n=1}^{m+1} b_{t,n}^2 \) is continuous with \( f(0) = A^2/m \) and \( f(1) = A^2/m \). By the intermediate value theorem, \( f \) takes all values in the interval \( [A^2/m, A^2/m] \). Hence, \( [A^2/m, A^2/m] \subseteq S \) for all \( m \in \mathbb{N} \); since the union of these intervals is \( (0, A^2] \), we get that \( (0, A^2] \subseteq S \).

4. Prove that

\[
\int_0^{\pi/2} \frac{(\cos x)^{\sin x} \sin x \, dx}{(\cos x)^{\sin x} + (\sin x)^{\cos x}} = \frac{\pi}{4}.
\]

Solution. Let \( I = \int_0^{\pi/2} \frac{(\cos x)^{\sin x} \sin x \, dy}{(\cos x)^{\sin x} + (\sin x)^{\cos x}} \). After the substitution \( y = \pi/2 - x \), we see that also \( I = \int_0^{\pi/2} \frac{(\sin x)^{\cos x} \cos x \, dy}{(\sin x)^{\cos x} + (\cos y)^{\sin x}} \). Then

\[
I + I = \int_0^{\pi/2} \frac{(\cos x)^{\sin x} \sin x \, dx}{(\cos x)^{\sin x} + (\sin x)^{\cos x}} + \int_0^{\pi/2} \frac{(\sin x)^{\cos x} \cos x \, dx}{(\sin x)^{\cos x} + (\cos x)^{\sin x}} = \int_0^{\pi/2} \frac{(\cos x)^{\sin x} + (\sin x)^{\cos x} \, dx}{(\cos x)^{\sin x} + (\sin x)^{\cos x}} = \int_0^{\pi/2} \frac{(\cos x)^{\sin x} + (\sin x)^{\cos x} \, dx}{(\cos x)^{\sin x} + (\sin x)^{\cos x}}.
\]

Hence, \( I = \pi/4 \).
5. Let \(ABCD\) be a convex quadrilateral satisfying \(|AB| = |CD|\). Let \(M\) and \(N\) be the midpoints of \(AD\) and \(BC\) respectively, and let the ray \(MN\) intersect the rays \(AB\) and \(DC\) at points \(P\) and \(Q\) respectively. Show that the angles \(\angle APN\) and \(\angle DNQ\) are equal.

**Solution.** Let \(O\) be the midpoint of the interval \(AC\). Then \(ON\) is a midline of \(\triangle ACB\), so is parallel to \(AB\), so \(\angle ONM = \angle APN\), and \(OM\) is a midline of \(\triangle CAD\), so is parallel to \(CD\), so \(\angle OMN = \angle DNQ\). Also, \(|ON| = \frac{1}{2}|AB| = \frac{1}{2}|CD| = |OM|\), so the triangle \(\triangle NOM\) is isosceles, and \(\angle OMN = \angle OMN\). Hence, \(\angle APN = \angle DNQ\).

Another solution. By the sine theorem for triangles, 
\[
\sin \frac{\angle DNQ}{\sin \frac{\angle CQN}{\sin \angle CQN}} = \sin \frac{\angle OCN}{\sin \angle CQN} \quad \text{and} \quad \frac{\angle DNQ}{\sin \frac{\angle CQN}{\sin \angle CQN}} = \frac{\angle OCN}{\sin \angle CQN}.
\]
Similarly, 
\[
\frac{\angle DNQ}{\sin \frac{AP}{\sin \angle APM}} = \sin \frac{\angle CQ}{\sin \angle CQN}.
\]
Since \(\angle DNQ = \angle OCN = \angle BNP\) and \(\sin \angle CQN = \sin \angle APM\), we get that 
\[
\sin \frac{\angle DNQ}{\sin \frac{\angle CQN}{\sin \angle CQN}} = \frac{\sin \angle APM}{\sin \angle APN}. \quad \text{Since} \quad |DC| = |AB|, \quad \text{this implies that} \quad |DNQ| = |APN|. \quad \text{So,} \quad \sin \angle DNQ = |DNQ| = |APN|.
\]
Also, we have \(\angle DNQ \geq \pi/2\) or \(\angle APN \geq \pi/2\); on either case \(|DNQ| = |APN| > |DNQ| = |APN|\), so both \(\angle APN\) and \(\angle DNQ\) are acute. Hence, \(\angle APN \sim \angle DNQ\).

6. Assume that three faces of a tetrahedron are pairwise orthogonal and have areas \(a, b, c\), and let the area of the fourth face be \(d\). Prove that \(d^2 = a^2 + b^2 + c^2\).

**Solution.** Let \(A, B, C, D\) be the vertices of the tetrahedron opposite to the faces whose areas are \(a, b, c, d\) respectively, let \(x = |DA|, y = |DB|, z = |DC|\), so that \(a = \frac{1}{2}yz, b = \frac{1}{2}xz, c = \frac{1}{2}xy\). Let \(E\) be the point of intersection of the segment \(AB\) with the plane containing \(DC\) and orthogonal to \(AB\).

Then \(|AB| = \sqrt{x^2 + y^2}\), \(|DE| = \frac{xy}{|AB|}\), \(|CE| = \sqrt{DE^2 + z^2}\), so
\[
d^2 = \frac{1}{4}|AB|^2|CE|^2 = \frac{1}{4}(x^2 + y^2)(x^2y^2/(x^2 + y^2) + z^2) = \frac{1}{4}(x^2y^2 + x^2z^2 + y^2z^2) = c^2 + b^2 + a^2.
\]

Another solution. Introduce the coordinate system such that the three orthogonal faces of the tetrahedron lay in the coordinate planes (and the common vertex of these faces is the origin of the coordinate system):

In this coordinate system, the other three vertices of the tetrahedron have coordinates \((x, 0, 0), (0, y, 0), (0, 0, z)\), so that \(a = \frac{xy}{2}, b = \frac{yz}{2}, c = \frac{xz}{2}\). The “area vector” of the fourth face is \(u = \frac{1}{2}(-x, y, 0) \times (-x, 0, z) = \frac{1}{2}(yz, xz, xy)\), thus \(d^2 = |u|^2 = \frac{1}{4}((yz)^2 + (xz)^2 + (xy)^2) = a^2 + b^2 + c^2\). 

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