Algebra Qualifying Exam I
2021

Directions

Put your name and the last four digits of your Social Security Number on the roster sheet when you receive it and enter a code name for yourself that is different from any code name that has already been entered.

Answer each question on a separate sheet or sheets of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

This is a closed book, closed notes exam.

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1. Let $p \geq 2$ be a prime and let G be a p–group (that is, $|G|$ is a power of p). Assume that G acts on a finite set X, and that $|X|$ is not divisible by p. Prove that the set of fixed points X^G is non–empty.

2. Prove that every finite, non–abelian, simple group must have a non–abelian, proper subgroup.

   Hint: Argue by contradiction. Show that any two maximal proper subgroups intersect trivially, and are conjugate to each other. Why is that absurd?

3. Let A be a unital, commutative (non-zero) ring. Consider the following sequence of A-linear maps, between A-modules
$$0 \to M' \longrightarrow M \longrightarrow M'' \to 0$$
   Prove that this sequence is exact if, and only if, for every maximal ideal $\mathfrak{m} \subsetneq A$, the following sequence of $A_{\mathfrak{m}}$-modules is exact
$$0 \to M'_{\mathfrak{m}} \longrightarrow M_{\mathfrak{m}} \longrightarrow M''_{\mathfrak{m}} \to 0$$
   Recall: for a maximal ideal (or more generally a prime ideal) $\mathfrak{m} \subsetneq A$, $A_{\mathfrak{m}}$ is the ring of fractions obtained by formally inverting elements not in $\mathfrak{m}$. That is, $A_{\mathfrak{m}} = S^{-1}A$, where $S = A \setminus \mathfrak{m}$.

4. Let R be a unital, commutative, finite (non-zero) ring. Prove that every prime ideal in R is maximal.

5. Let V be a finite-dimensional vector space over a field K, and let $n = \dim(V)$. Let $T : V \to V$ be a K–linear map and let $A = (a_{ij})_{1 \leq i,j \leq n}$ be the matrix representing T relative to a basis $\{v_1, \ldots, v_n\}$ of V.

   Let $1 \leq r \leq n$ and $\wedge^r V$ be the r-th exterior power of V. Determine the matrix of $\wedge^r T : \wedge^r V \to \wedge^r V$.