Algebra Qualifying Exam II
2021

Directions

Put your name and the last four digits of your Social Security Number on the roster sheet when you receive it and enter a code name for yourself that is different from any code name that has already been entered.

Answer each question on a separate sheet or sheets of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

This is a closed book, closed notes exam.

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1. Let $I = \{1, 2, 3, \ldots\}$ together with the usual order. Consider the following inverse system on I valued in the category of abelian groups, denoted by $\mathfrak{Z} = (\{Z_n\}_{n \in I}; \{\varphi_{nm}\}_{n \leq m})$:

   - $Z_n = \mathbb{Z}$ for every $n \in I$.
   - For every $n \leq m$, the group homomorphism $\varphi_{nm} : Z_m \to Z_n$ is given by $\varphi_{nm}(x) = 3^{m-n}x$.

   Prove that $\varprojlim_{(I,\leq)} \mathfrak{Z} = (0)$.

2. Let R be an integral domain and M be an R–module. Assume that M is divisible and torsion–free. Prove that M is injective.

   If, in addition R is a principal ideal domain, show that M is injective if, and only if it is divisible.

3. Let k be a field and $R = k[x, y]$. Consider the following R–modules: $M = R/(y)$ and $N = R/(y - x^2)$. For each $n \geq 0$, compute $\mathrm{Tor}_n(M, N)$.

4. State Galois correspondence theorem for finite field extensions. Give an example of an infinite Galois extension, where this correspondence fails, if the Krull topology on the Galois group is not taken into consideration.

5. Let
$$f(x) = x^3 - ax + b \in \mathbb{Z}[x]$$
   be a cubic polynomial with integer coefficients such that its discriminant $4a^3 - 27b^2$ is a perfect square. Assume that $\gcd(a, b) > 2$ and is square–free (i.e, for every prime p, p^2 does not divide the $\gcd(a, b)$). Prove that $f(x)$ is irreducible and its splitting field is a cyclic cubic extension of $\mathbb{Q}$.